2.3 Real Zeroes of Polynomial Functions

Next, we are interested in how to get the factored form of higher degree polynomials so that we can get the real zeros fast. We will review, factoring and long division to divide polynomials. Then we will use that to factor higher degree polynomial and interpret the remainder (if any).

Example 1: Divide the polynomial $f(x) = x^3 - 3x^2 + 4$ by $x - 2$ and use the result to factor the polynomial completely.

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

where $r(x) = 0$ *or* the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, then $d(x)$ *divides evenly* into $f(x)$.

The division algorithm can also be written as

Example 2: Divide $f(x) = (x^4 - 5x^2 + 4)$ by $(x^2 - 1)$.

Example 3: Divide $f(x) = -2 + 3x - 5x^2 + 4x^3 + 2x^4$ by $x^2 + 2x - 3$

Practice:

1. Divide the polynomial $f(x) = (x^3 - 9x^2 + 26x - 24)$ by $d(x) = (x - 2)$.
   (a) Does $x - 2$ divide evenly in $f(x)$?
   (b) Calculate the value of $f(2)$.

2. Divide $f(x) = (x^4 - 5x^3 + 2x^2 + 8x - 6)$ by $d(x) = (x - 3)$
   (a) Does $x - 3$ divide evenly in $f(x)$?
   (b) Calculate the value of $f(3)$.

The Remainder and Factor Theorem

Remainder Theorem:
If a polynomial $f(x)$ is divided by $x - a$, then the remainder is equal to $f(a)$.
Factor Theorem:
A polynomial $f(x)$ has a factor $(x - a)$ if and only if $f(a) = 0$.

The Rational Zero Test

What if we are not given any roots or zeros of the polynomial but we are still asked to factor it. For example, in Example 1 we were told to divide $f(x) = x^3 - 3x^2 + 4x$ by $x - 2$ and in doing that we were able to completely factor out $f(x)$. Next, we are interested in how to find potential zeros. This is the result of The Rational zero Test.

The Rational Zero Test

If the polynomial

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$

has integer coefficients, then every rational zero of f has the form

Rational zero $= \frac{p}{q}$

where p and q have no common factors other than 1, p is a factor of the constant term a_0, and
q is a factor of the leading coefficient a_n.

Possible rational zeros =

Example 4: Find the rational zeros of

1. $f(x) = x^3 - 3x^2 + 4x$.

2. $h(x) = 2x^3 + 3x^2 - 8x + 3$

3. $g(x) = x^3 + x + 1$

A nth degree polynomial can have at most n real roots.
In the complex number system , every nth degree polynomial has exactly n zeros.

Recall Complex Numbers

$\sqrt{-15} =$
$\sqrt{-9} =$
A complex number is a number of the form $a + bi$, where a and b are constants.
a is
b is
EX $2 + 3i$

EX $\sqrt{5}i$

EX -17

Note: -17 is a **both** a complex number **and** a real number.
Note Every real number is a complex number, but not every complex number is a real number.

Types of Zeros of a polynomial function f

- Real

- Complex

Fundamental Theorem of Algebra

Every polynomial function with degree $n \geq 1$ can be factored into n linear factors of the following form:
$f(x) = a(x - r_1) \cdot (x - r_2) \cdot (x - r_3) \cdot ... \cdot (x - r_n)$
where $r_1, r_2, r_3, ..., r_n$ are complex numbers and $a \neq 0$.

Example 5: Completely factor the polynomial over the complex numbers.
$f(x) = (x - 1)(x^2 + 4)(x^2 - 3)$

Example 6: List all possible rational zeros. Find all zeros and completely factor f over the complex numbers.
$f(x) = 2x^4 + x^3 + 17x^2 + 9x - 9$