

1. Transformations of functions

In this section, we'll discuss some ways to graph more complicated popular functions. For example, we will find a quick way to graph the function $f(x) = -(x + 2)^2 + 5$ just by knowing that this graph is the graph of x^2 (which we call the **parent** or **basic** function) transformed in some way. These transformation can be rigid or non-rigid.

A **rigid transformation** changes the location of the function in a coordinate plane, but leaves the size and shape of the graph unchanged.

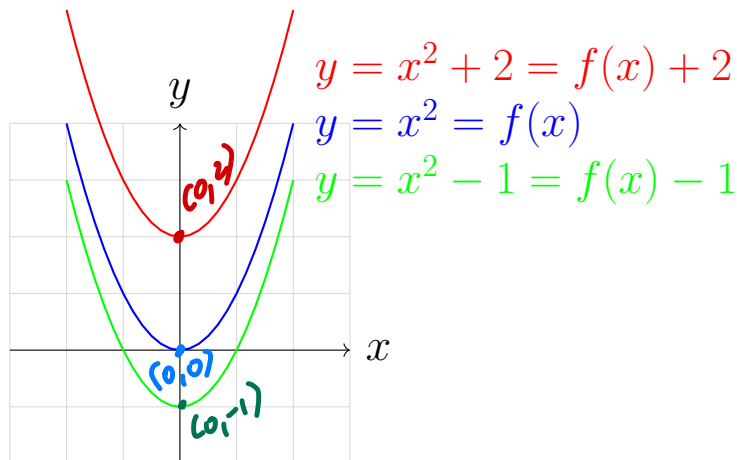
A **non-rigid transformation** changes the size and/or shape of the graph.

1 Shifting

Vertical Shifting (UP or Down)

For any function $f(x)$, adding or subtracting a constant k shifts the graph:

- $f(x) + k$ shifts the graph up k units $x^2 + 2, \quad x^2 - 5$
- $f(x) - k$ shifts the graph down k units



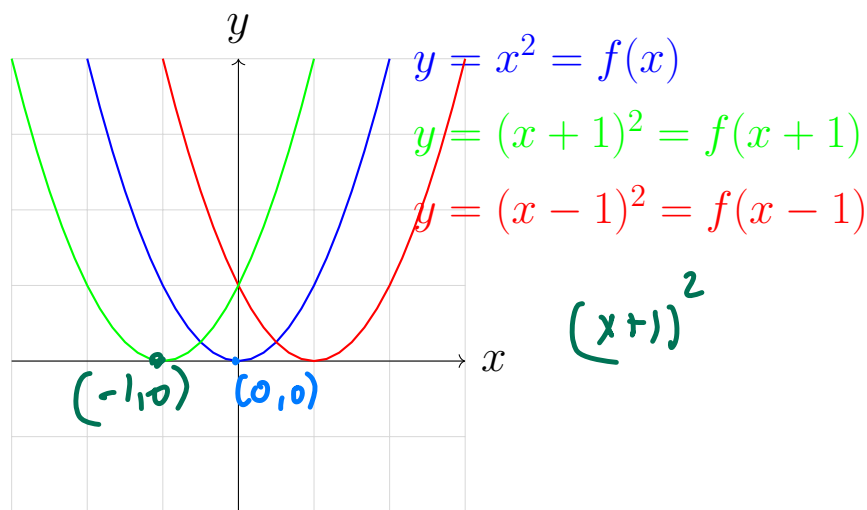
Horizontal Shifting (LEFT or RIGHT)

$$x^2 \rightarrow (x+2)^2 \text{ or } (x-3)^2$$

For any function $f(x)$, replacing x with $(x \pm h)$ shifts the graph:

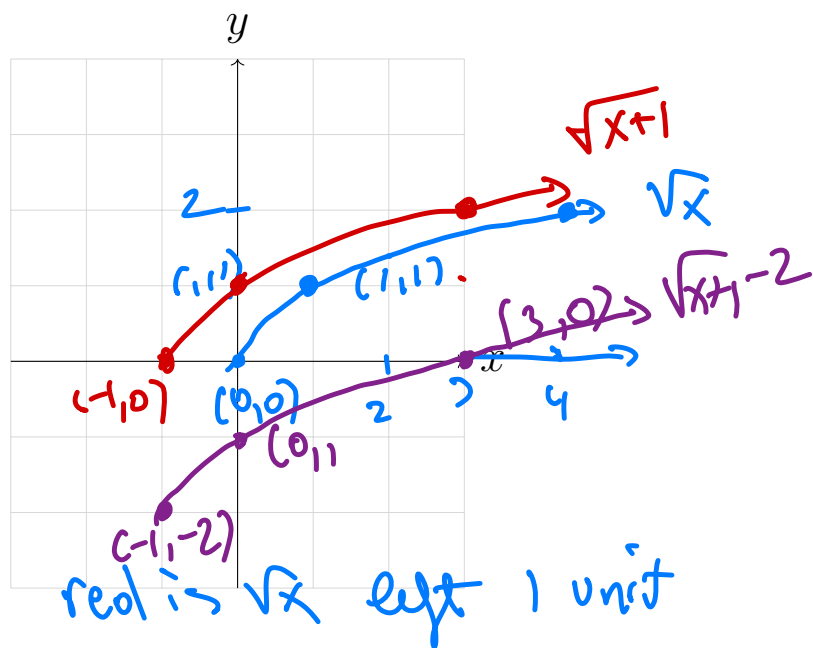
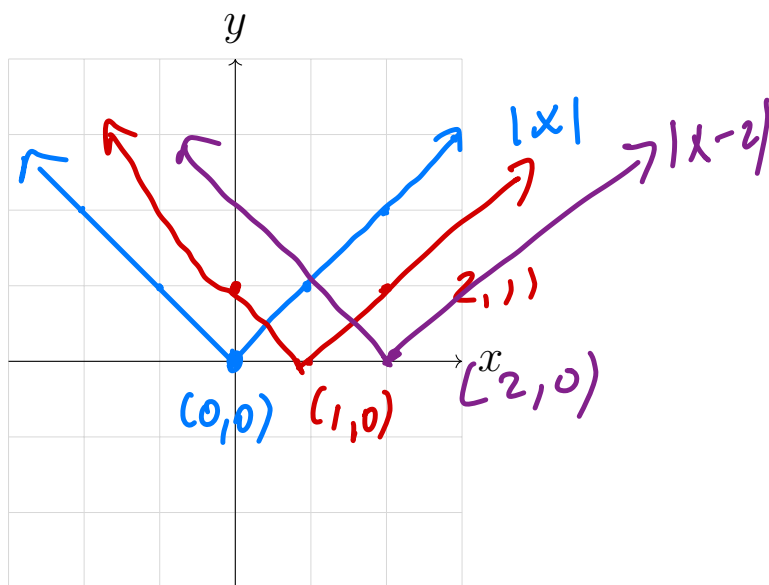
- $f(x - h)$ shifts the graph right h units $(x-3)^2$ right 3 units
- $f(x + h)$ shifts the graph left h units $(x+2)^2$ left 2 units

Note: The shift is in the opposite direction of the sign inside the parentheses!



Example 1. Describe the transformation and sketch the graph of each function:

1. $f(x) = |x - 2|$ parent function $|x|$ right 2 units $|x-2|$
2. $f(x) = \sqrt{x+1} - 2$



2 Reflecting

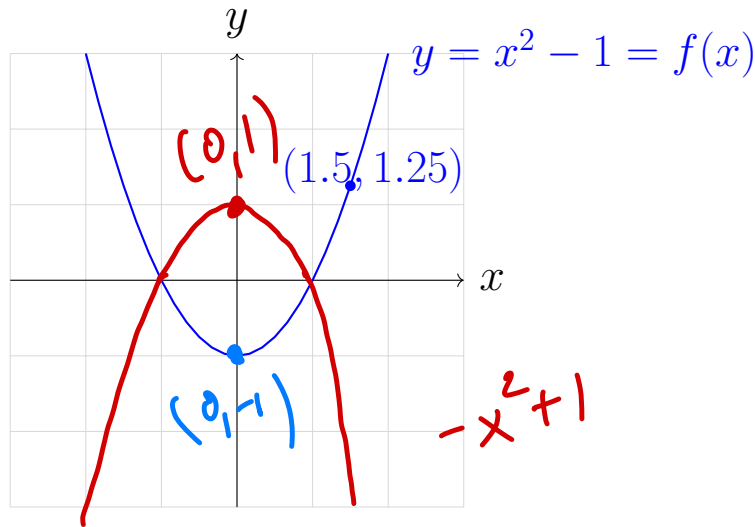
Reflecting over x-axis

For any function $f(x)$, multiplying by -1 reflects the graph over the x-axis:

$$\rightarrow -x^2, -\sqrt{x+5}, -|x|$$

- $y = -f(x)$ reflects the graph of $y = f(x)$ over the x-axis
- Each point (x, y) becomes $(x, -y)$

$$y = -\sqrt{x}$$

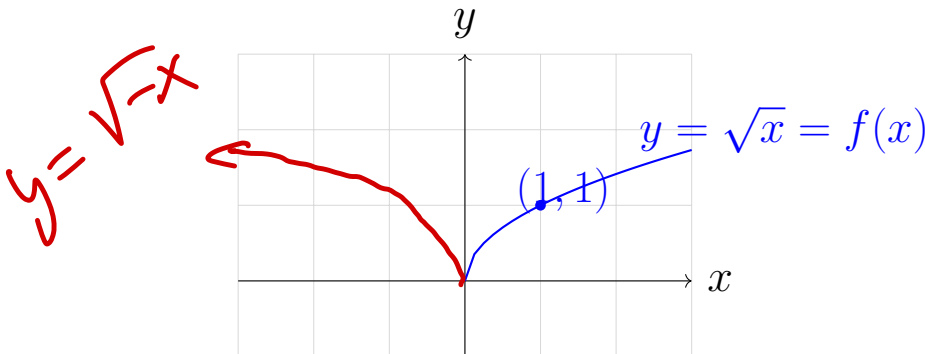


Reflection over the y-axis

$$(-x)^2, \sqrt{-x}, | -x + 1 |$$

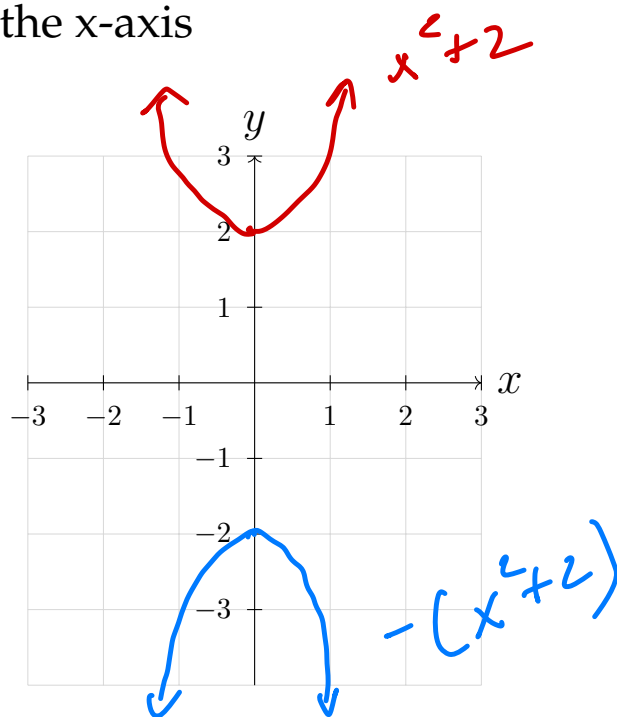
For any function $f(x)$, replacing x with $-x$ reflects the graph over the y-axis:

- $y = f(-x)$ reflects the graph of $y = f(x)$ over the y-axis
- Each point (x, y) becomes $(-x, y)$

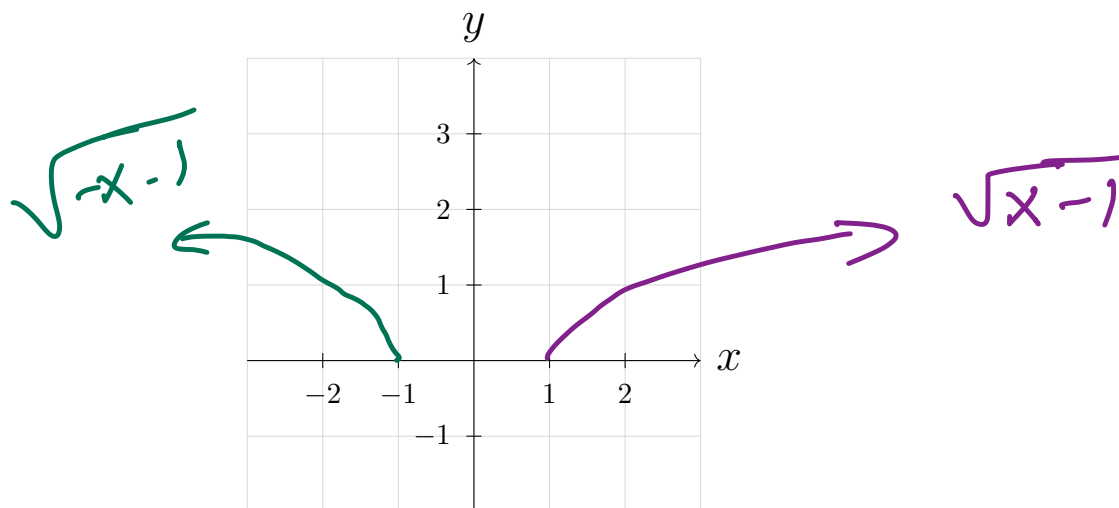


Example 2. Find the equation and sketch the graph of each reflection:

1. Reflect $y = x^2 + 2$ over the x-axis



2. Reflect $y = \sqrt{x - 1}$ over the y-axis



$$\begin{aligned} f(x) &= \sqrt{x-1} \\ f(-x) &= \sqrt{-x-1} \end{aligned}$$

3 Stretching (non-rigid transformation)

$$(2x)^2 \text{ or } |2x| \\ \text{or } \sqrt{1/3x}$$

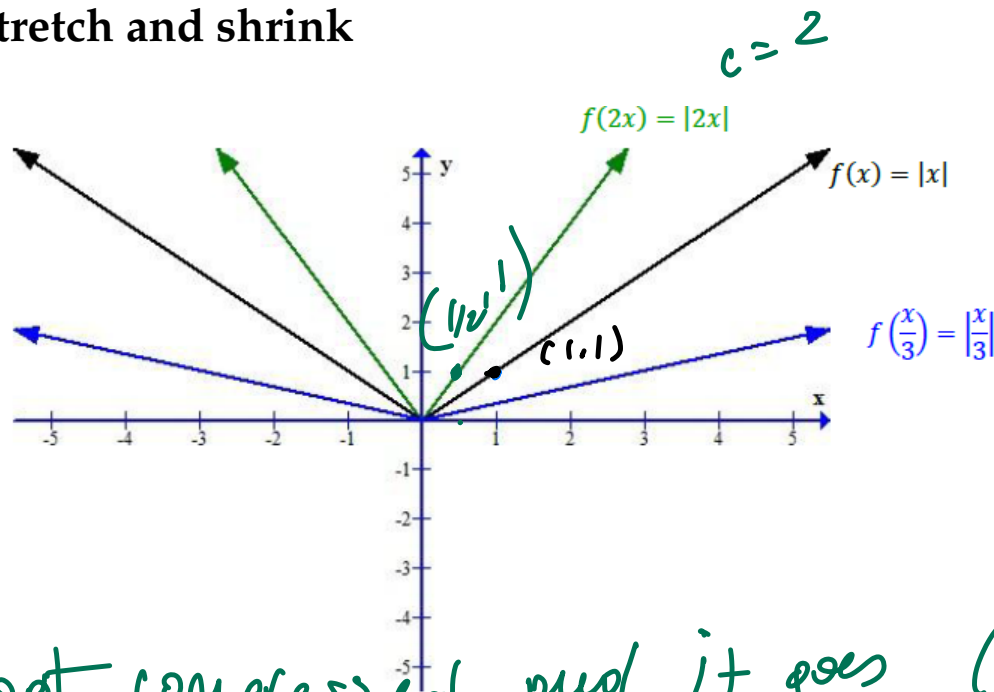
Horizontal stretch/shrink

For any function $f(x)$, if c is multiplied to the variable of the function then the graph of the function will undergo a horizontal stretching or compression.

- When the function becomes $y = f(cx)$ and $0 < c < 1$, a horizontal stretching of the graph of will occur.
- Graphically, a horizontal stretching pulls the graph of $y = f(x)$ away from the y -axis.
- When $c > 1$ in the function $y = f(cx)$, a horizontal shrinking of the graph of $y = f(x)$ will occur.
- A horizontal shrinking pushes the graph of toward the y -axis.
- In general, a horizontal stretching or shrinking means that every point (x, y) on the graph of is transformed to $(\frac{x}{c}, y)$ on the graph of $y = f(cx)$.

Example 3.

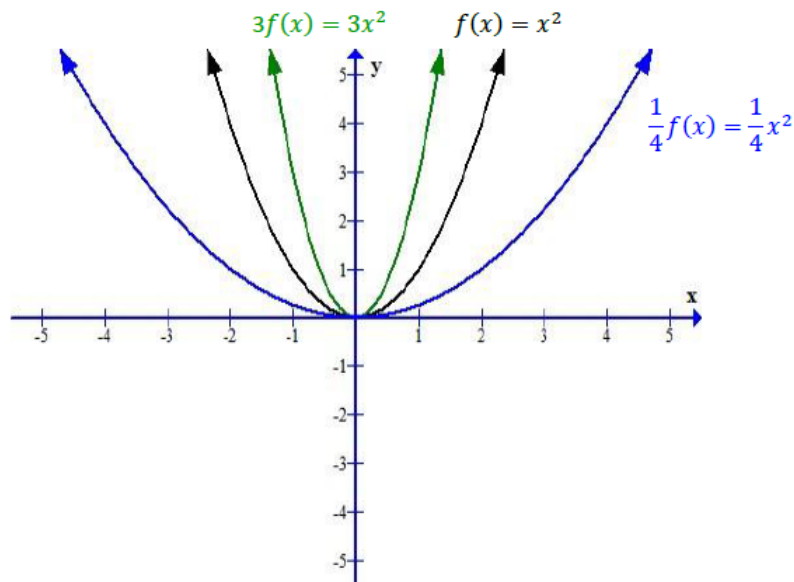
Horizontal Stretch and shrink



$(1, 1)$ get compressed and it goes $(\frac{x}{c}, y)$
 $(\frac{1}{2}, 1)$

Vertical stretching and shrinking: non-rigid

- If c is multiplied to the function then the graph of the function will undergo a vertical stretching or compression.
- So when the function becomes $y = cf(x)$ and $0 < c < 1$, a vertical shrinking of the graph of will occur.
- Graphically, a vertical shrinking pulls the graph of $y = f(x)$ toward the x -axis.
- When $c > 1$ in the function $y = cf(x)$, a vertical stretching of the graph of $y = f(x)$ will occur.
- A vertical stretching pushes the graph of $y = f(x)$ away from the x -axis.
- In general, a vertical stretching or shrinking means that every point (x, y) on the graph of $f(x)$ is transformed to (x, cy) on the graph of $y = cf(x)$.



4 Using transformations to graph functions

Transformations can be combined within the same function so that one graph can be shifted, stretched, and reflected. If a function contains more than one transformation, **perform the transformations in the following order** to graph the function:

1. Horizontal translation

(left/right) $(x-2)$, $(x+3)$
 $|x-2|$,

2. Stretching or shrinking

3. Reflecting

4. Vertical translation

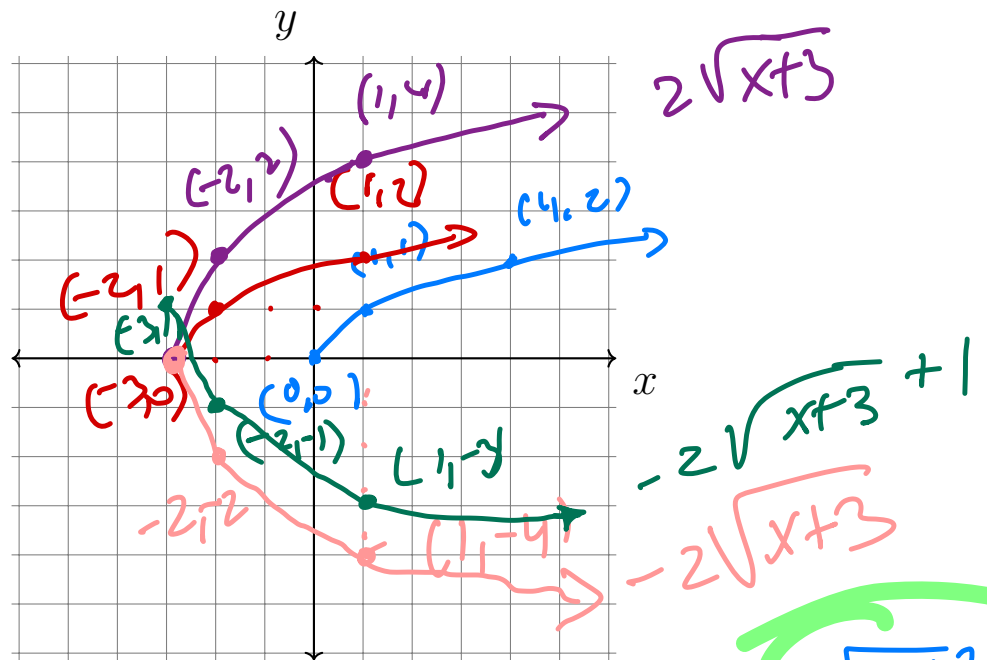
Summary

Transformations of the graphs of functions	
$f(x) + c$	shift $f(x)$ up c units
$f(x) - c$	shift $f(x)$ down c units
$f(x + c)$	shift $f(x)$ left c units
$f(x - c)$	shift $f(x)$ right c units
$f(-x)$	reflect $f(x)$ about the y-axis
$-f(x)$	reflect $f(x)$ about the x-axis
$cf(x)$	When $0 < c < 1$ – vertical shrinking of $f(x)$ When $c > 1$ – vertical stretching of $f(x)$ Multiply the y values by c
$f(cx)$	When $0 < c < 1$ – horizontal stretching of $f(x)$ When $c > 1$ – horizontal shrinking of $f(x)$ Divide the x values by c

Practice: Graph the parent function $\sqrt{x}$ and the function

$$f(x) = -2\sqrt{x+3} + 1.$$

Where are the points $(0, 0)$, $(1, 1)$, and $(4, 2)$ on the graph of $\sqrt{x}$ transformed to on the graph of $-2\sqrt{x+3} + 1$?



1) Horiz. trans. yes left 3 units

2) stretch or shrink yes
vertical stretch $(x, y) \rightarrow (x, 2y)$

$$(-2, 1) \rightarrow (-2, 2)$$

$$(1, 2) \rightarrow (1, 4)$$

3) Reflection yes over x-axis $-2\sqrt{x+3}$

4) up 2 unit