Mth 146: Day 4
7.3 Trig Substitution

1 Part 1: New Concepts & Worked Examples

Integrals involving radicals can be difficult to deal with directly. But often it is possible to use a substitution to eliminate the radical entirely and work with a much simpler integral. This is the point of trig. substitution.

The goal is to get a perfect square under the radical using the Pythagorean identities. The initial form of the radicand determines which substitution should be used. To remember each type, it's helpful to connect it with a corresponding Pythagorean identity:

1. $\sqrt{a^2 - x^2}$: (think $1 - \sin^2 * = \cos^2 *$)
   $\implies$ substitute $x = a\sin(\theta)$, $dx = a\cos(\theta)$

2. $\sqrt{a^2 + x^2}$: (think $1 + \tan^2 * = \sec^2 *$)
   $\implies$ substitute $x = a\tan(\theta)$, $dx = a\sec^2(\theta)$

3. $\sqrt{x^2 - a^2}$: (think $\sec^2 * - 1 = \tan^2 *$)
   $\implies$ substitute $x = a\sec(\theta)$, $dx = a\sec(\theta)\tan(\theta)$

In general, the most common mistakes tend to be either forgetting the dx term or just simple algebra. After the substitution, all the usual techniques for trig. integrals can still be used. At the end, don't forget to rewrite the final integral in terms of the original variable. (Drawing a right triangle is helpful here.)

Note: For more complicated integrals, you may need to perform a u-sub or complete the square to bring the integral into a form in which a trig. sub. can be used.

Example 1. $I = \int \frac{x^3}{\sqrt{x^2+64}}\,dx$

$\sqrt{a^2+x^2}$ think $1+\tan^2 x = \sec^2 x$

Step 1: Rewrite your radical and choose the right substitution.

$\sqrt{x^2+64} = \sqrt{64+x^2} = \sqrt{8^2+x^2}$

$x = 8\tan\theta$

$dx = 8\sec^2\theta\,d\theta$

Step 2: Rewrite the integral in terms of θ.

$$I = \int \frac{8^3\tan^3\theta}{\sqrt{8^2+8^2\tan^2\theta}}\,8\sec^2\theta\,d\theta$$

Step 3: Eliminate the radical using the right Pythagorean identity.

$$I = \int \frac{8^3\tan^3\theta}{\sqrt{8^2(1+\tan^2\theta)}}\,8\sec^2\theta\,d\theta = \int \frac{8^3\tan^3\theta}{\sqrt{8^2\sec^2\theta}}\,8\sec^2\theta\,d\theta$$

$$= \int \frac{8^3\tan^3\theta}{8\sec\theta}\,8\sec^2\theta\,d\theta = \int 8^3\tan^3\theta\sec\theta\,d\theta$$

Step 4: Solve the resulting integral using known techniques (u-sub, trig integral, algebra).

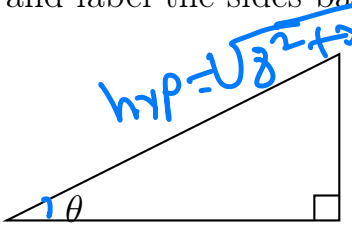
$$I = \int 8^3\tan^2\theta\tan\theta\sec\theta\,d\theta = \int 8^3(u^2-1)\,du$$

$u = \sec\theta$

$du = \sec\theta\tan\theta\,d\theta$

$$= 8^3\left(\frac{u^3}{3} - u\right) + C = 8^3\left(\frac{\sec^3\theta}{3} - \sec\theta\right)$$

Step 5: Use right-triangle trig to write the final answer in terms of x. Draw a right triangle below, label the angle θ, and label the sides based on your substitution.



need $\sec\theta = ?$

$\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{8^2+x^2}}{8}$

$\text{adj}^2 + \text{opp}^2 = \text{hyp}^2$

$x = 8\tan\theta$

$\frac{x}{8} = \tan\theta$

$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{x}{8}$

$\text{hyp}^2 = 8^2+x^2$

$\text{hyp} = \sqrt{8^2+x^2}$

Final answer

$$I = \frac{8^3}{3}\left(\frac{\sqrt{8^2+x^2}}{8}\right)^3 - 8^3\frac{\sqrt{8^2+x^2}}{8}$$

$$= \frac{\left(\sqrt{8^2+x^2}\right)^3}{3} - 8^2\sqrt{8^2+x^2} + C$$

Example 2. $I = \int \frac{1}{(x^2-1)^{3/2}}\,dx$

$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

$(x^2-1)^{3/2} = \sqrt{(x^2-1)^3} = (\sqrt{x^2-1})^3$

Step 1: Rewrite your radical and choose the right substitution.

$I = \int \frac{1}{(\sqrt{x^2-1})^3}\,dx$

$x = \sec\theta$

$dx = \sec\theta\tan\theta\,d\theta$

Step 2: Rewrite the integral in terms of θ.

$I = \int \frac{1}{(\sqrt{\sec^2\theta - 1})^3}\sec\theta\tan\theta\,d\theta$

$\frac{1}{\sin^2\theta} = \frac{1}{u^2} = u^{-2}$

$\sec^2\theta - 1 = \tan^2\theta$

Step 3: Eliminate the radical using the right Pythagorean identity.

$I = \int \frac{1}{(\sqrt{\tan^2\theta})^3}\sec\theta\tan\theta\,d\theta = \int \frac{1}{(\tan\theta)^3}\sec\theta\tan\theta\,d\theta$

Step 4: Solve the resulting integral using known techniques (u-sub, trig integral, algebra).

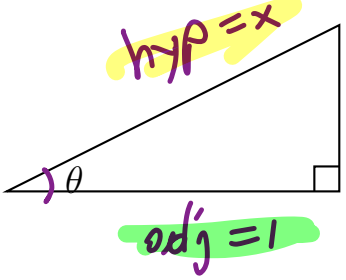
$I = \int \frac{\sec\theta\tan\theta}{\tan^2\theta}\,d\theta = \int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta$

$= \int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int u^{-2}\,du = \frac{u^{-1}}{-1} = -\frac{1}{u} = -\frac{1}{\sin\theta} = -\csc\theta$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

Step 5: Use right-triangle trig to write the final answer in terms of x. Draw a right triangle below, label the angle θ, and label the sides based on your substitution.



$x = \sec\theta$

$\sec\theta = \frac{x}{1} = \frac{hyp}{adj}$

$adj^2 + opp^2 = hyp^2$

$1^2 + opp^2 = x^2$

$opp^2 = x^2 - 1$

$opp = \sqrt{x^2-1}$

$I = -\csc\theta = -\frac{hyp}{opp} = \boxed{-\frac{x}{\sqrt{x^2-1}} + C}$

Example 3. $I = \int \frac{x}{\sqrt{3-2x-x^2}}\,dx$

complete the square

$3-2x-x^2 = -(x^2+2x-3)$

$= -(x^2+2x+1-1-3)$

$= -[(x+1)^2 - 4]$

$= 4 - (x+1)^2$

$= 2^2 - (x+1)^2$

Step 1: Rewrite your radical and choose the right substitution.

$\sqrt{3-2x-x^2} = \sqrt{2^2-(x+1)^2}$

$x+1 = 2\sin\theta$

constant^2 − variable^2

Step 2: Rewrite the integral in terms of θ. $dx = 2\cos\theta\,d\theta$

$$I = \int \frac{(2\sin\theta - 1)\,2\cos\theta\,d\theta}{\sqrt{2^2-(2\sin\theta)^2}} = \int \frac{2\sin\theta - 1}{\sqrt{2^2(1-\sin^2\theta)}}\,2\cos\theta\,d\theta =$$

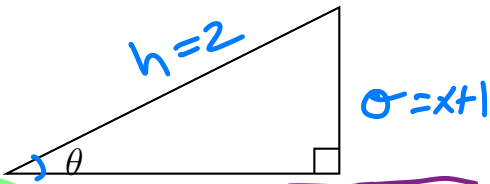
Step 3: Eliminate the radical using the right Pythagorean identity.

$$I = \int \frac{2\sin\theta - 1}{\sqrt{2^2\cos^2\theta}}\,2\cos\theta\,d\theta = \int \frac{2\sin\theta - 1}{2\cos\theta}\,2\cos\theta\,d\theta = \int 2\sin\theta - 1\,d\theta$$

Step 4: Solve the resulting integral using known techniques (u-sub, trig integral, algebra).

$$I = 2\int \sin\theta\,d\theta - \int 1\,d\theta = -2\cos\theta - \theta + C$$

Step 5: Use right-triangle trig to write the final answer in terms of x. Draw a right triangle below, label the angle θ, and label the sides based on your substitution.



$x+1 = 2\sin\theta$

$\frac{x+1}{2} = \sin\theta = \frac{\theta}{h}$

To get θ

$\sin\theta = \frac{x+1}{2}$

$\theta = \arcsin\left(\frac{x+1}{2}\right)$

$a^2 + \theta^2 = h^2$

$a^2 + (x+1)^2 = 2^2$

$a^2 = 2^2 - (x+1)^2$

$a = \sqrt{2^2-(x+1)^2}$

$\cos\theta = \frac{a}{h} = \frac{\sqrt{2^2-(x+1)^2}}{2}$

$I = -2\cos\theta - \theta + C$

$= -2\,\frac{\sqrt{2^2-(x+1)^2}}{2} - \arcsin\frac{x+1}{2} + C$

2 Part 2: Your Turn — Practice

Try the following individually or in small groups. Final answers are given as footnotes at the bottom of the page — only the final answer, not the steps, so use them to check your work rather than replace it.

1. $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$.[1]

$x = 4\sin\theta$
$dx = 4\cos\theta\,d\theta$

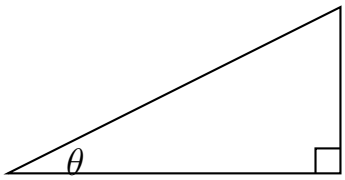
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Step 2: Rewrite the integral in terms of θ.

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Step 5: Use right-triangle trig to write the final answer in terms of x. Draw a right triangle below, label the angle θ, and label the sides based on your substitution.



[1] Answer: $8\arcsin\left(\frac{x}{4}\right) - \frac{x\sqrt{16-x^2}}{2} + C$.

2. $\int \sqrt{13+12x-x^2}\,dx$.[2]

See next page for solution

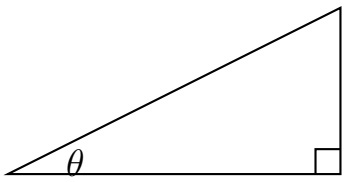
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Step 4: Solve the resulting integral using known techniques (u-sub, trig integral, algebra).

Step 5: Use right-triangle trig to write the final answer in terms of x. Draw a right triangle below, label the angle θ, and label the sides based on your substitution.



[2] Answer: $\frac{x-6}{2}\sqrt{13+12x-x^2}+\frac{49}{2}\arcsin\left(\frac{x-6}{7}\right)+C$.

Practice

$1-\sin^2 x = \cos^2 x$

$1-\cos^2 x = \sin^2 x$

1. $\int \sqrt{13+12x-x^2}\,dx$

2. $\int \frac{x^2}{\sqrt{16-x^2}}\,dx$ use: $x = 4\sin\theta$

① complete square:

$$13+12x-x^2 = 13+\left(\frac{12}{2}\right)^2-\left(\frac{12}{2}\right)^2+12x-x^2$$

$$= 13+36-36+12x-x^2$$

$$= 49-(36-12x+x^2)$$

$$= 49-(x-6)^2$$

$$\int \sqrt{13+12x-x^2}\,dx = \int \sqrt{49-(x-6)^2}\,dx$$

$x-6 = 7\sin\theta$

$dx = 7\cos\theta\,d\theta$

$$= \int \sqrt{49-(7\sin\theta)^2}\cdot 7\cos\theta\,d\theta$$

$$= \int \sqrt{49-49\sin^2\theta}\;7\cos\theta\,d\theta$$

$$= \int \sqrt{49(1-\sin^2\theta)}\;7\cos\theta\,d\theta$$

$$= \int \sqrt{49(\cos^2\theta)}\;7\cos\theta\,d\theta$$

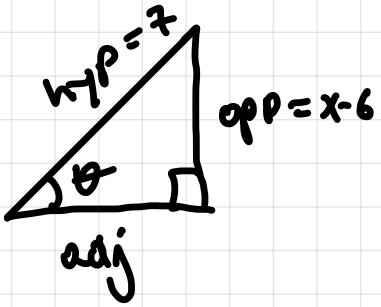
$$= \int 7\cos\theta\;7\cos\theta\,d\theta$$

$$= 49\int \cos^2\theta\,d\theta \longrightarrow \text{power reducing....}$$

$$= 49\int \frac{1+\cos(2\theta)}{2}\,d\theta$$

$$= 49 \int \frac{1+\cos(2\theta)}{2}\, d\theta$$

$$= 49\left[\int \frac{1}{2}\, d\theta + \frac{1}{2}\int \cos(2\theta)\, d\theta\right] = 49\left[\frac{\theta}{2} + \frac{1}{2}\cdot\frac{\sin(2\theta)}{2}\right]$$



$x-6 = 7\sin\theta$

$\sin\theta = \frac{x-6}{7} = \frac{opp}{hyp}$

$adj = \sqrt{7^2-(x-6)^2}$

$adj^2 + opp^2 = hyp^2$

$adj^2 = 7^2 - (x-6)^2$

To solve for θ $\quad \arcsin(\sin\theta) = \arcsin\left(\frac{x-6}{7}\right)$

$$\theta = \arcsin\left(\frac{x-6}{7}\right)$$

$\sin(2\theta)$ in terms of x

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{x-6}{7}\right)\left(\frac{\sqrt{49-(x-6)^2}}{7}\right)$$

Integral $= 49\left[\frac{\arcsin\left(\frac{x-6}{7}\right)}{2} + \frac{1}{2}\cdot\frac{2\left(\frac{x-6}{7}\right)\sqrt{49-(x-6)^2}}{7}\right]$

3 Part 3: Reflection

For each learning outcome below, rate how comfortable you feel right now, and write a few sentences: What clicked? What's still fuzzy? What would help?

1. **Recognize situations where replacing a variable with a trigonometric function can help to simplify a given integral.**

   Comfort level (circle one): 1 — not comfortable 2 3 4 5 — very comfortable

   Reflection:

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2. **Know how to use a reference triangle for help in reversing trigonometric substitutions and in simplifying trigonometric expressions.**

   Comfort level (circle one): 1 — not comfortable 2 3 4 5 — very comfortable

   Reflection:
