Mth 146: Day 8
7.8 Improper Integrals

1 Part 1: New Concepts & Worked Examples

So far we have examined definite integrals only for continuous functions $f(x)$ over finite intervals $[a, b]$, since those are the conditions under which the Fundamental Theorem of Calculus applies.

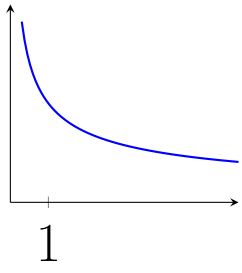
Our next goal is to examine integrals

- over an unbounded interval

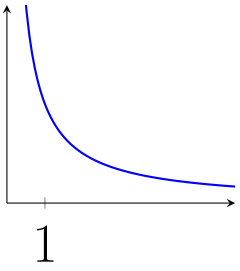
- where the integrand is an unbounded function (the function is discontinuous somewhere in the interval and it goes to ∞ or $-\infty$ at that point)

These types of integral are called **improper**.

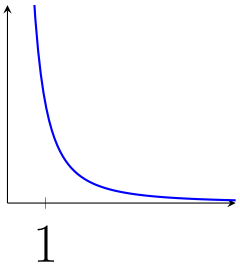
Here are the graphs of $\frac{1}{\sqrt{x}}$, $\frac{1}{x}$, and $\frac{1}{x^2}$ on $[1, \infty)$. As you look at each one, think about whether the area under the curve on this interval appears to be finite or infinite:



$\frac{1}{\sqrt{x}}$



$\frac{1}{x}$



$\frac{1}{x^2}$

Algebraically:

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$$

$$\int_1^\infty \frac{1}{x}\,dx$$

$$\int_1^\infty \frac{1}{x^2}\,dx$$

Suppose a and b are any real numbers and $f(x)$ is a function.

1. If f is continuous on $[a, \infty)$, then

2. If f is continuous on $(-\infty, b]$, then

3. If f is continuous on $(-\infty, \infty)$, then

In all three cases, if the limits involved exist, then we say that the improper integral **converges** to the value determined by those limits, and if the limits are infinite or fail to exist, then we say that the improper integral **diverges**.

Example 1.

For what values of p is the integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ convergent?

Example 2. Determine whether the improper integral converges and, if so, evaluate it.

$$\int_1^\infty x^{-3/2}\,dx$$

Example 3. Determine whether the improper integral converges and, if so, evaluate it.

$$\int_0^\infty xe^{-x^2}\,dx$$

Example 4. Determine whether the improper integral converges and, if so, evaluate it.

$$\int_1^\infty \frac{\ln(x)}{x}\,dx$$

Example 5. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

Try the following individually or in small groups. The final answer is given as a footnote at the bottom of the page — only the final answer, not the steps, so use it to check your work rather than replace it.

1. $\displaystyle\int_{-\infty}^{-2} \frac{2}{x^2-1}\,dx.$[1]

[1] Answer: The integral converges to $\ln 3$.

3 Part 3: Reflection

For each learning outcome below, rate how comfortable you feel right now, and write a few sentences: What clicked? What's still fuzzy? What would help?

1. **Define and compute (if possible) integrals of the form**

$$\int_a^\infty f(x)\,dx, \qquad \int_{-\infty}^b f(x)\,dx, \qquad \int_{-\infty}^\infty f(x)\,dx.$$

Comfort level (circle one): 1 — not comfortable 2 3 4 5 — very comfortable

Reflection:
