Mth 146: Day 10
Sequences and Series Review

1 Why sequences and series?

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

Much of Calculus II is about one idea: *adding up infinitely many numbers* and deciding whether the total makes sense. This is how calculators compute e^x, $\sin x$, and $\ln x$ (Taylor series), and how we approximate functions with polynomials. Today we review the precalculus language of sequences and series and then add the calculus tool that makes everything work: the **limit**.

2 Sequences and notation

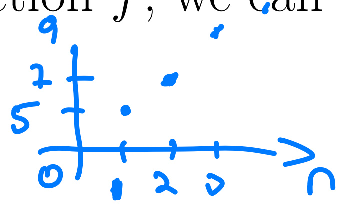
Roughly speaking, a sequence is an infinite list of numbers written in a definite order.

Definition. A **sequence** is a function whose domain is the set of positive inters $1, 2, 3, \ldots n, n+1, \ldots$ (sometimes $\{0, 1, 2, \ldots\}$). The numbers in the list are called the terms of the sequence.

Notation:

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots \qquad \{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}$$

a_n is called the n^{th} or the general term. Since $a_n = f(n)$ for a function f, we can *graph* a sequence as isolated dots (n, a_n).



Example 1. Find the first three terms and the 100th term. Then write the sequence using { } notation.

(a) $a_n = 2n - 1$

$a_1 = 2 \cdot 1 - 1 = 1$

$a_2 = 2 \cdot 2 - 1 = 3$

$a_3 = 2 \cdot 3 - 1 = 5$

$a_{100} = 2 \cdot 100 - 1 = 199$

$\{1, 3, 5, \ldots, 199, \ldots\}$

(b) $b_n = n^2 - 1$

$b_1 = 0$

$b_2 = 3$

$b_3 = 8$

$b_{100} = 9999$

(c) $c_n = \dfrac{n}{n+1}$

$c_1 = \frac{1}{2}$

$c_2 = \frac{2}{3}$

$c_3 = \frac{3}{4}$

$c_{100} = \frac{100}{101}$

(d) $d_n = \dfrac{(-1)^{n+1}}{n}$

$d_1 = 1$

$d_2 = -\frac{1}{2}$

$d_3 = 1/3$

$d_4 = -\frac{1}{4}$

$d_{100} = \frac{-1}{100}$

Example 2. Find a formula for the general term a_n, assuming the pattern continues. (Start at $n = 1$.)

(a) $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \cdots$

$a_n = \frac{n+1}{n+2}$

(c) $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \cdots$

$c_n = \frac{1}{n!}$

(b) $-2, 4, -8, 16, -32, \cdots$

$b_n = (-2)^n$

$n=1 \quad b_1 = (-2)^1 = -2$

$n=2 \quad b_2 = (-2)^2 = 4$

$n=3 \quad b_3 = (-2)^3 = -8$

(d) $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \cdots$

$d_n = (-1)^{n+1} 2^{-n} = \frac{(-1)^{n+1}}{2^n}$

Definition. An **alternating sequence** is a sequence whose ________________.

Tip: $(-1)^n$ makes the __odd__-numbered terms negative;
$(-1)^{n+1}$ makes the __even__-numbered terms negative.

Recursively defined sequences. When the nth term depends on some or all of the terms before it, the sequence is __defined recursevely__.

Example 3. Find the first five terms: $a_1 = 4$ and $a_n = 2a_{n-1} + 1$ for $n > 1$.

$a_1 = 4$

$a_2 = 2a_1 + 1 = 2 \cdot 4 + 1 = 9$

$a_3 = 2a_2 + 1 = 2 \cdot 9 + 1 = 19$

$a_4 = 2 \cdot 19 + 1 = 39$

$\{4, 9, 19, 39, \ldots\}$

3 Factorials

Factorials appear in Taylor series and are the key to the Ratio Test. The skill you need in Calc II is *simplifying* ratios of factorials.

Definition. For a positive integer n,
$n! =$ $1 \cdot 2 \cdot 3 \cdots n = 1 \cdot 2 \cdot 3 \cdots (n-1)(n)$,
and by definition $0! =$ 1.
$5! =$ $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

Key identities: $(n+1)! =$ $1 \cdot 2 \cdot 3 \cdots n \cdot (n+1)$

$(2n+2)! =$ $1 \cdot 2 \cdot 3 \cdots (2n)(2n+1)(2n+2)$

Example 4. Simplify.

(a) $\dfrac{8!}{3!\cdot 5!} = \dfrac{\cancel{5!}\cdot \cancel{6}\cdot 7\cdot 8}{\cancel{3!}\cdot \cancel{5!}} = 56$

$8! = \cancel{3!\,5!}$

$8! = \underbrace{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot \underline{6\cdot 7\cdot 8}$

$8! = 5!\,6\cdot 7\cdot 8$

(b) $\dfrac{n!}{(n+2)!} = \dfrac{\cancel{n!}}{\cancel{n!}\,(n+1)(n+2)} = \dfrac{1}{(n+1)(n+2)}$

$(n+2)! = \underbrace{1\cdot 2\cdots n}_{n!}(n+1)(n+2)$

$\cancel{(n+2)! = n! + 2!}$

(c) $\dfrac{(2n+2)!}{(2n)!} = \dfrac{\cancel{1\cdot 2\cdot 3\cdots (2n)}(2n+1)(2n+2)}{\cancel{(2n)!}}$

$= (2n+1)(2n+2)$

(d) $\dfrac{(n+1)!}{(n+1)^{n+1}}\cdot\dfrac{n^n}{n!}$ (a Ratio Test preview)

$= \dfrac{\cancel{n!}\,(n+1)\cdot n^n}{(n+1)^{n+1}\,\cancel{n!}} = \dfrac{(n+1)\,n^n}{(n+1)^{n+1}} =$

$= \dfrac{\cancel{(n+1)}\,n^n}{(n+1)^n\,\cancel{(n+1)}} = \left(\dfrac{n}{n+1}\right)^n$

This is the new, Calc II part. What happens to a_n as $n \to \infty$?

Definition. We write $\lim_{n\to\infty} a_n = L$ if we can make the terms a_n as close to L as we like by taking n sufficiently large. If the limit exists (and is a finite number) the sequence converges; otherwise it diverges.

$\{a_n\} = \{\frac{n}{n+1}\}$

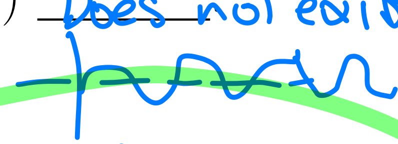
$\lim_{n\to\infty} \frac{n}{n+1} = 1$ $(L=1)$

Theorem (sequences from functions). If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ for every integer n, then $\lim_{n\to\infty} a_n = L$.

$\lim_{x\to\infty} \frac{1}{x} = 0$ then $\lim_{n\to\infty} \frac{1}{n} = 0$

Why it matters: we can use all of our Calc I tools on sequences, including L'Hopital's Rule.

Warning: the converse is false. $a_n = \sin(\pi n) = 0$ for all n, so $a_n \to 0$, but $\lim_{x\to\infty} \sin(\pi x)$ Does not exist.



$a_1 = \sin \pi = 0$
$a_2 = \sin 2\pi = 0$
$a_3 = \sin 3\pi = 0$

A useful fact: $\lim_{n\to\infty} \frac{1}{n^r} = 0$ if $r > 0$.

Limit Laws for Sequences. Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant. Then

1. **Sum Law**

   $$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

2. **Difference Law**

   $$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

3. **Constant Multiple Law**

   $$\lim_{n\to\infty} c\,a_n = c \lim_{n\to\infty} a_n$$

4. **Product Law**

   $$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$$

5. **Quotient Law**

   $$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

6. **Power Law**

   $$\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

Squeeze Theorem. If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim a_n = \lim c_n = L$, then $\lim b_n = L$.

$a_n \le b_n \le c_n$

$\downarrow \quad \downarrow \quad \downarrow$

$L \quad L \quad L$

Corollary. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n =$ 0.

$-|a_n| \le a_n \le |a_n|$

$-0 = 0 \qquad \downarrow 0 \qquad \downarrow 0$

Continuity. If $\lim_{n\to\infty} a_n = L$ and f is continuous at L, then $\lim_{n\to\infty} f(a_n) =$ $f(L)$.

$r = 1/3$

$\left(\frac{1}{3}\right)^n = \frac{1}{3^n}$

A limit you will use constantly:

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \text{diverges} & \text{if } r \leq -1 \text{ or } r > 1 \end{cases}$$

So $\{r^n\}$ converges exactly when $-1 < r \le 1$.

$r = (-1/2)$

Example 5. Determine whether each sequence converges or diverges. If it converges, find the limit.

divide by n^2

(a) $a_n = \dfrac{3n^2 - n}{5n^2 + 4}$

$$\lim_{n\to\infty} \frac{3n^2-n}{5n^2+4}$$

$$= \lim_{n\to\infty} \frac{\frac{3n^2}{n^2} - \frac{n}{n^2}}{\frac{5n^2}{n^2} + \frac{4}{n^2}} = \lim_{n\to\infty} \frac{3 - \frac{1}{n}}{5 + \frac{4}{n^2}}$$

$\frac{1}{n} \to 0$, $\frac{4}{n^2} \to 0$

$$= 3/5$$

(b) $a_n = \dfrac{\ln n}{n}$

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

Let $f(x) = \dfrac{\ln x}{x}$

$$\lim_{x\to\infty} \frac{\ln x}{x} \overset{l'H}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0$$

(c) $a_n = \dfrac{(-1)^n}{n}$

$$\lim_{n\to\infty} \frac{(-1)^n}{n}$$