Mth 146: Day 16
11.8 Power Series

1 Part 1: New Concepts & Worked Examples

Power Series

So far we have studied series of numbers: $\sum a_n$. Now we consider series in which each term includes a power of the variable x.

Definition. A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = _____$$

where x is a _____ and the c_n's are constants called the _____ of the series.

For each number we substitute for x, the power series is a series of constants that we can test for convergence or divergence. A power series may converge for some values of x and diverge for others. The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

whose domain is _____.
Notice that f resembles a polynomial — the only difference is that f has _____.

Example: the geometric series. If we take $c_n = 1$ for all n, the power series becomes

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

which converges when _____ and diverges when _____.

- At $x = \frac{1}{2}$: _____. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n =$ _____

- At $x = 2$: _____. $\displaystyle\sum_{n=0}^{\infty} 2^n =$ _____

Definition. More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = \text{_____}$$

is called a **power series in** $(x - a)$, or a _____ (or a **power series about** a).

Notes:

- For the $n = 0$ term we use the convention $(x - a)^0 = 1$, even when $x = a$.
- When $x = a$, all of the terms are 0 for $n \geq 1$, so a power series _____.

Key idea: To find the values of x for which a power series converges, we normally use the _____ (or the Root Test).

Example 1. For what values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

Example 2. For what values of x is the series $\sum_{n=0}^{\infty} n!\, x^n$ convergent?

Example 3. For what values of x does the series $\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$ converge?

Radius and Interval of Convergence

For every power series we have looked at so far, the set of x-values where the series converges has turned out to be an interval. This is true in general.

Theorem. For a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

(i) The series converges _____.

(ii) The series converges _____.

(iii) There is a positive number R such that the series _____ and _____.

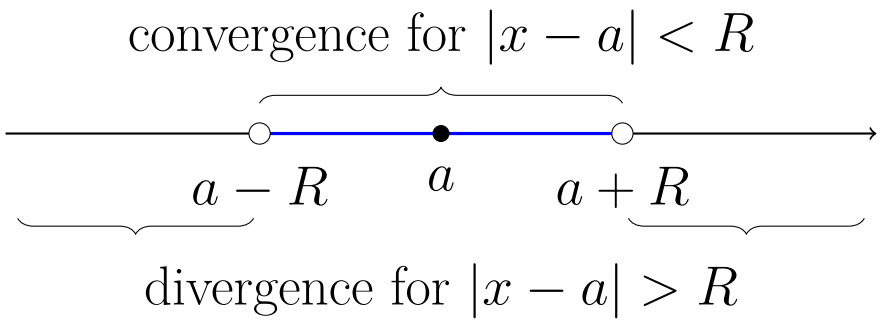
Definitions.

The number R in case (iii) is called the _____ of the power series. By convention, $R =$ _____ in case (i) and $R =$ _____ in case (ii).

The _____ of a power series is the interval that consists of _____.

- In case (i), the interval is the single point _____.
- In case (ii), the interval is _____.
- In case (iii), $|x-a| < R$ can be rewritten as _____.

The endpoints. When x is an *endpoint*, $x = a \pm R$, anything can happen — the series might converge at one or both endpoints, or diverge at both. So in case (iii) there are four possibilities for the interval of convergence:



Summary of the examples so far:

	Series	**Radius**	**Interval**
Geometric	$\sum_{n=0}^{\infty} x^n$		
Example 1	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$		
Example 2	$\sum_{n=0}^{\infty} n!\, x^n$		
Example 3	$\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$		

Note: Use the Ratio Test (or sometimes the Root Test) to find the radius of convergence R. The Ratio and Root Tests **always fail** when x is an endpoint of the interval of convergence, so _____.

Strategy for finding R and the interval of convergence:

Step 1. Apply the Ratio Test: compute $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, which will depend on x.
Step 2. Set that limit < 1 and solve for $|x - a| < R$. This gives the radius R.
Step 3. Plug each endpoint $x = a - R$ and $x = a + R$ into the series and test the resulting series of numbers (Test for Divergence, p-series, Comparison, AST, ...).
Step 4. Write the interval, using [ ] or ( ) at each endpoint.

Example 4. Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}.$$

Example 5. Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}.$$

2 Part 2: Now You Try It

Find the **radius of convergence** and **interval of convergence** of each series. Final answers are given as footnotes at the bottom of the page — only the final answer, not the steps, so use them to check your work rather than replace it.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^n n\, x^n$ [1]

2. $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{2n-1}$ [2]

[1] Answer: $R = 1$; interval $(-1, 1)$.
[2] Answer: $R = 1$; interval $[-1, 1)$.

3. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ [3]

4. $\displaystyle\sum_{n=1}^{\infty} n^n x^n$ [4]

[3] Answer: $R = \infty$; interval $(-\infty, \infty)$.
[4] Answer: $R = 0$; interval $\{0\}$.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2 x^n}{2^n}$ [5]

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1}$ [6]

[5] Answer: $R = 2$; interval $(-2, 2)$.
[6] Answer: $R = 1$; interval $[1, 3]$.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$ [7]

[7] Answer: $R = \frac{5}{2}$; interval $[-2, 3)$.

3 Part 3: Reflection

For each item below, rate your comfort level and jot a quick note: What clicked? What's still fuzzy? What would help?

1. **Recognize a power series $\sum c_n(x-a)^n$ and identify its center a and coefficients c_n.** 1 2 3 4 5

2. **Use the Ratio Test to find the values of x where a power series converges.** 1 2 3 4 5

3. **Find the radius of convergence R, including the cases $R = 0$ and $R = \infty$.** 1 2 3 4 5

4. **Test the endpoints with another test (Test for Divergence, p-series, Comparison, AST).** 1 2 3 4 5

5. **Write the interval of convergence with the correct brackets at each endpoint.** 1 2 3 4 5