MTH 122: Practice Final Exam

Total Possible Points: 150

Name: ________________________________ Date: ________________

Instructions: For each section, you will be asked to reflect on the learning outcome being tested. Write any relevant formulas you will use and/or a short paragraph explaining your understanding of the concept. Then solve the problem(s).

Learning Outcome

Learning Outcome 1: Evaluate sequences and series.

Reflection

Before solving the problems below, write the formulas you will need to evaluate sequences and series. Include formulas for arithmetic and geometric sequences/series.

Problem 1. **(3 points)**

The nth term of a sequence is given. Find the indicated term. Express your answer in simplest form.

$$b_n = \frac{2n}{(n+3)!}; \quad \text{find } b_5.$$

Problem 2. **(4 points)**

Find the sum of the geometric series, if possible.

$$\sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{k-1}$$

A. The sum does not exist.

B. The sum does exist. The sum is __________. Express the sum as an integer or simplified fraction.

Problem 3. **(3 points)**

Find the sum.

$$7 + 11 + 15 + \cdots + 71$$

Learning Outcome

Learning Outcome 2: Determine sequence formulas from given sequence element values.

Reflection

Explain in a short paragraph how you identify patterns in sequences to determine the general term a_n. What strategies do you use for arithmetic vs. geometric sequences?

Problem 4. **(4 points)**

Find the nth term a_n of a sequence whose first four terms are given.

$$-\frac{9}{4}, \quad -\frac{10}{8}, \quad -\frac{11}{12}, \quad -\frac{12}{16}, \quad \ldots$$

Learning Outcome

Learning Outcome 3: Express a given series using sigma notation.

Reflection

Write a short paragraph explaining what sigma notation represents and how to convert between expanded form and sigma notation.

Problem 5. **(3 points)**

Write the following series using sigma notation:

$$7 + 11 + 15 + 19 + 23 + 27 + 31$$

Learning Outcome

Learning Outcome 6: Identify the domain and range of functions and relations.

Reflection

List the key restrictions to consider when finding domains (e.g., division by zero, square roots, logarithms). Also explain how to find range from a graph.

Problem 6. **(8 points)**

Consider the following piecewise defined function.

$$f(x) = \begin{cases} -x^2 + 4 & \text{for } x \leq 2 \\ 3x & \text{for } x > 2 \end{cases}$$

(a) Graph the function.

(b) Write the domain in interval notation.

(c) Write the range in interval notation.

(d) Evaluate $f(-2)$, $f(2)$, and $f(3)$.

(e) Find the value(s) of x for which $f(x) = 9$.

Learning Outcome

Learning Outcome 7: Demonstrate an understanding of function graphs, their transformations, and their properties.

Reflection

Write the transformation rules for $f(x+h)$, $f(x)+k$, $-f(x)$, $f(-x)$, and $af(x)$. Also define what it means for a function to be even or odd.

Problem 7. **(3 points)**

The graph of $y = f(x)$ is given. Graph the function $y = f(-x) + 1$.

Problem 8. **(3 points)**

Determine whether the function is even, odd, or neither.

$$g(x) = \sqrt{4 + x^2}$$

A. The function g is an even function.

B. The function g is an odd function.

C. The function g is neither even nor odd.

Learning Outcome

Learning Outcome 8: Construct and analyze the composition of functions.

Reflection

Write the definition of $(f \circ g)(x)$ and $(g \circ f)(x)$. Explain why composition of functions is generally not commutative.

Problem 9. **(5 points)**

Given $f(x) = x^2 + 8x$, find $\dfrac{f(x+h) - f(x)}{h}$ and simplify.

Problem 10. **(4 points)**

Given $k(x) = -5x + 2$ and $m(x) = \dfrac{1}{x}$,

(a) Find $(k \circ m)(1)$.

(b) Find $(m \circ k)(2)$.

Learning Outcome

Learning Outcome 9: Create and analyze graphs of polynomial functions.

Reflection

Write the vertex formula for a parabola $f(x) = ax^2 + bx + c$. Explain how to determine if a parabola opens upward or downward and how to find its maximum or minimum value.

Problem 11. **(3 points)**

Find the coordinates of the vertex of the parabola by applying the vertex formula.

$$f(x) = 2x^2 - 16x + 119$$

$(h, k) =$ ________

Problem 12. **(4 points)**

The gas mileage $m(x)$ (in mpg) for a certain vehicle can be approximated by

$$m(x) = -0.033x^2 + 2.637x - 35.013$$

where x is the speed of the vehicle in mph.

(a) Determine the speed at which the car gets its maximum mileage. Round your answer to the nearest mph.

(b) Determine the maximum gas mileage. Round your answer to one decimal place.

Learning Outcome

Learning Outcome 10: Use appropriate theorems and techniques to determine the zeros and factors of polynomial functions.

Reflection

List the theorems you know for finding zeros of polynomials (e.g., Rational Zero Theorem, Factor Theorem, Complex Conjugate Theorem). Explain how to use synthetic division.

Problem 13. **(5 points)**

A polynomial $f(x)$ and three of its zeros are given.

$$f(x) = 3x^5 + 17x^4 + 48x^3 + 24x^2 - 131x + 39$$

$-2 - 3i$ and $\frac{1}{3}$, -3 are zeros.

(a) Is 1 a zero?

(b) Find all the zeros. Write the answer in exact form.

(c) Factor $f(x)$ as a product of linear factors.

Learning Outcome

Learning Outcome 11: Create and analyze graphs of rational functions.

Reflection

Explain how to find vertical asymptotes, horizontal asymptotes, and slant asymptotes of a rational function. Include the rules for horizontal asymptotes based on degree comparison.

Problem 14. **(5 points)**

For the graph of

$$f(x) = \frac{(3x+2)(x+6)}{(x+1)(3x+5)}$$

(a) Identify the x-intercepts.

(b) Identify any vertical asymptotes.

(c) Identify the horizontal asymptote or slant asymptote if applicable.

(d) Identify the y-intercept.

Learning Outcome

Learning Outcome 12: Recognize when an inverse function exists, and form the inverse when possible.

Reflection

Explain what it means for a function to be one-to-one. Describe the Horizontal Line Test and the process for finding an inverse function algebraically.

Problem 15. **(5 points)**

A one-to-one function is given. Write an equation for the inverse function.

$$t(x) = \frac{x-3}{x+4}$$

Learning Outcome

Learning Outcome 13: Create and analyze graphs of exponential and logarithmic functions.

Reflection

Write the properties of logarithms (product rule, quotient rule, power rule). Also write the change of base formula.

Problem 16. **(3 points)**

Write the logarithm as a sum or difference of logarithms. Simplify each term as much as possible.

$$\log\left(\frac{10{,}000}{\sqrt{p^2+q^2}}\right)$$

Problem 17. **(3 points)**

Write the logarithmic expression as a single logarithm with coefficient 1, and simplify as much as possible. Assume that the variables represent positive real numbers.

$$5\log_7 m - 7\log_7 n - 6\log_7 p$$

Learning Outcome

Learning Outcome 14: Solve exponential and logarithmic equations.

Reflection

Write the relationship between exponential and logarithmic forms:

$$\log_b(x) = y \Leftrightarrow b^y = x$$

. Explain strategies for solving exponential and logarithmic equations.

Problem 18. **(3 points)**

Solve the equation. Write the solution set with the exact values given in terms of common or natural logarithms. Also give approximate solutions to at least 4 decimal places.

$$6e^{3n-2} + 3 = 15$$

Problem 19. **(3 points)**

Solve the equation. Write the solution set with the exact solutions. Make sure to verify the solutions.

$$\log_3(q) - 3 = -\log_3(q+6)$$

Learning Outcome

Learning Outcome 18: Apply the algebraic and graphical techniques learned in this class to solve application problems.

Reflection

Write the compound interest formulas: $A = P\left(1+\frac{r}{n}\right)^{nt}$ **and** $A = Pe^{rt}$**. Also write the exponential decay model** $Q = Q_0e^{-kt}$**.**

Problem 20. **(5 points)**

Use the model $A = Pe^{rt}$ or $A = P\left(1+\dfrac{r}{n}\right)^{nt}$, where A is the future value of P dollars invested at interest rate r compounded continuously or n times per year for t years.

A \$10,000 investment grows to \$12,507.51 at 4.5% interest compounded quarterly. For how long was the money invested? Round to the nearest year.

Problem 21. **(8 points)**

A sample of radioactive material decays according to the model $Q = Q_0 e^{-kt}$, where Q is the amount of radioactive material (in grams) present at time t (in years), Q_0 is the initial amount, and k is the decay constant. The initial amount of the radioactive sample is 50 grams and it decays at a rate of 3.5% per year.

(a) How much of the original 50-gram sample will remain after 30 years? Express your answer in grams to the nearest hundredth.

(b) Find the half-life of this radioactive material. The half-life is the time required for half of any given amount of the material to decay.

(c) After how many years will only 5 grams of the original sample remain? Express your answer to the nearest year.

Problem 22. **(6 points)**
A lawn service company charges \$51 for each lawn maintenance call. The fixed monthly cost of \$657 includes telephone service and depreciation of equipment. The variable costs include labor, gasoline, and taxes and amount to \$36 per lawn.

(a) Write a linear cost function representing the monthly cost $C(x)$ for x maintenance calls.

(b) Write a linear revenue function representing the monthly revenue $R(x)$ for x maintenance calls.

(c) Write a linear profit function representing the monthly profit $P(x)$ for x maintenance calls.

(d) Determine the number of lawn maintenance calls needed per month for the company to make money. Round up your answer to the nearest maintenance call.

(e) If 41 maintenance calls are made for a given month, how much money will the lawn service make or lose? Write your answer to the nearest cent.

Learning Outcome

Learning Outcome 15: Solve linear and non-linear systems of equations.

Reflection

List the methods for solving systems of equations (substitution, elimination, graphing). Explain when you would use each method.

Problem 23. **(4 points)**

Solve the system of equations by using any method.

$$\begin{cases} 6x - 3y = -3 \\ 5x + 18y = -23 \end{cases}$$

The solution set is __________.

Problem 24. **(4 points)**

Solve the following system of nonlinear equations.

$$\begin{cases} x^2 + y^2 = 25 \\ y = x + 1 \end{cases}$$

Learning Outcome

Learning Outcome 16: Perform operations on matrices using matrix algebra.

Reflection

Explain when matrix addition/subtraction is possible and when matrix multiplication is possible. What are the dimension requirements for each operation?

Problem 25. **(8 points)**

Consider the following matrices.

$$A = \begin{bmatrix} 2 & 5 & 0 \\ -3 & 1 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 6 \\ 6 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}$$

(a) What is the size of matrix A?

(b) What is the element a_{23}?

(c) Find $A + B$ if possible.

(d) Find $A - C$ if possible.

(e) Find AB if possible.

(f) Find $3C$ if possible.

Learning Outcome

Learning Outcome 17: Solve systems of linear equations using matrices.

Reflection

Describe row-echelon form and reduced row-echelon form. List the elementary row operations. Also write the formula for finding the inverse of a 2×2 matrix.

Problem 26. **(2 points)**

Write the augmented matrix for the given system.

$$\begin{cases} 6x + 4y - 6z = 9 \\ 9x + 7z = 3 \\ 2y - 9z = 7 \end{cases}$$

Problem 27. **(2 points)**

Determine if the matrix is in row-echelon form. If not, explain why.

$$\begin{bmatrix} 1 & 2 & -4 & 7 \\ 0 & 1 & -5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 28. **(4 points)**

Solve the system by using Gaussian elimination or Gauss-Jordan elimination.

$$\begin{cases} 2x = -71 - 9y - 4z \\ x + 5y = 2z - 39 \\ 9z = 17 - x + 3y \end{cases}$$

The solution set is __________.

Problem 29. **(2 points)**
An augmented matrix is given. Determine the number of solutions to the corresponding system of equations.

$$\begin{bmatrix} 1 & 7 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

A. The system has one solution.

B. The system has no solution.

C. The system has infinitely many solutions.

Problem 30. **(2 points)**
An augmented matrix is given. Determine the number of solutions to the corresponding system of equations.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

A. The system has one solution.

B. The system has no solution.

C. The system has infinitely many solutions.

Problem 31. **(3 points)**
Are the following two 2×2 matrices inverses of each other? Show your work.

$$A = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ -\frac{5}{2} & \frac{3}{2} \end{bmatrix}$$

Problem 32. **(3 points)**

Find the inverse of the following 2×2 matrix:

$$C = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$$

Problem 33. **(5 points)**

Solve the following 3×3 system of linear equations using inverse matrices:

$$\begin{cases} 2x + y - z = 3 \\ x + 3y + 2z = 11 \\ 3x - y + z = 2 \end{cases}$$

1. Write down the coefficient matrix A and the constants matrix B.
2. Find the inverse of the coefficient matrix. A^{-1}
3. Give the solution to the system.