

$$f(x) = (x+1)(x-2)(x+3)$$

$x=1 \quad x=2 \quad x=-3$

Day 18 Notes

Real Zeros of Polynomial Functions

Real Zeros of Polynomial Functions

Next, we are interested in how to get the factored form of higher degree polynomials so that we can get the real zeros fast. We will review, factoring and long division to divide polynomials. Then we will use that to factor higher degree polynomial and interpret the remainder (if any).

Example 1: Divide the polynomial $f(x) = x^3 - 3x^2 + 4$ by $x - 2$ and use the result to factor the polynomial completely.

$$\begin{array}{r}
 \text{divisor } x-2 \overline{) \begin{array}{l} \text{quotient } x^2 - x - 2 \\ x^3 - 3x^2 + 0x + 4 \\ \underline{x^3 - 2x^2} \\ 0 - x^2 + 0x + 4 \\ \underline{-x^2 + 2x} \\ 0 - 2x + 4 \\ \underline{-2x + 4} \\ 0 \end{array} } \\
 \text{remainder } 0
 \end{array}$$

$$\begin{aligned}
 \frac{x^3}{x} &= x^2 \\
 x^2(x-2) &= x^3 - 2x^2 \\
 -3x^2 - (-2x^2) &= -3x^2 + 2x^2 \\
 -3x^2 + 2x^2 &= -x^2
 \end{aligned}$$

$$\begin{aligned}
 \frac{-x^2}{x} &= -x \\
 -x(x-2) &= -x^2 + 2x
 \end{aligned}$$

$$\frac{-2x}{x} = -2$$

$$\begin{aligned}
 -2(x-2) &= -2x + 4
 \end{aligned}$$

$$f(x) = x^3 - 3x^2 + 4 = (x-2)(x^2 - x - 2)$$

$$= (x-2)(x-2)(x+1)$$

$$f(x) = (x-2)^2(x+1)$$

$x=2$ zero $m=2$
 $x=-1$ zero $m=1$

touches
crosses

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

Diagram labels:
 - $f(x)$ is labeled "dividend"
 - $d(x)$ is labeled "divisor"
 - $q(x)$ is labeled "quotient"
 - $r(x)$ is labeled "remainder"
 - A red arrow points from the remainder term $r(x)$ to the expression $\div d(x)$.

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, then $d(x)$ divides evenly into $f(x)$.

The division algorithm can also be written as

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

Example 2: Divide $f(x) = (x^4 - 5x^2 + 4)$ by $(x^2 - 1)$.

$$\begin{array}{r}
 x^2 - 4 \\
 x^2 - 1 \overline{) \begin{array}{l} x^4 + 0x^3 - 5x^2 + 0x + 4 \\ \underline{x^4} - x^2 + 4 \\ - 4x^2 + 0x + 4 \\ \underline{-4x^2 + 4} \\ 0 \end{array}
 \end{array}$$

$$f(x) = (x^2 - 1)(x^2 - 4) = (x - 1)(x + 1)(x - 2)(x + 2)$$

Example 3: Divide $f(x) = -2 + 3x - 5x^2 + 4x^3 + 2x^4$ by $x^2 + 2x - 3$

$$\begin{array}{r}
 2x^2 + 1 \\
 \hline
 x^2 + 2x - 3 \overline{) 2x^4 + 4x^3 - 5x^2 + 3x - 2} \\
 \underline{2x^4 + 4x^3 - 6x^2} \\
 x^2 + 3x - 2 \\
 \underline{x^2 + 2x - 3} \\
 x + 1
 \end{array}$$

$\approx$

$$a) f(x) = d(x) g(x) + r(x)$$

or

$$b) \frac{f(x)}{d(x)} = g(x) + \frac{r(x)}{d(x)}$$

$$a) 2x^4 + 4x^3 - 5x^2 + 3x - 2 = (x^2 + 2x - 3)(2x^2 + 1) + x + 1$$

$$b) \frac{2x^4 + 4x^3 - 5x^2 + 3x - 2}{x^2 + 2x - 3} = 2x^2 + 1 + \frac{x + 1}{x^2 + 2x - 3}$$

* useful in Calc 2

Practice:

1. Divide the polynomial $f(x) = (x^3 - 9x^2 + 26x - 24)$ by $d(x) = (x - 2)$.

(a) Does $x - 2$ divide evenly in $f(x)$? yes

(b) Calculate the value of $f(2)$.

$$x^2 - 7x + 12$$

$$\begin{array}{r} x-2 \overline{) x^3 - 9x^2 + 26x - 24} \\ \underline{x^3 - 2x^2} \\ -7x^2 + 26x - 24 \\ \underline{-7x^2 + 14x} \\ 12x - 24 \\ \underline{12x - 24} \\ 0 \end{array}$$

$r=0$

$$f(2) = 2^3 - 9 \cdot 2^2 + 26 \cdot 2 - 24 = 8 - 36 + 52 - 24 = 0$$

2. Divide $f(x) = (x^4 - 5x^3 + 2x^2 + 8x - 6)$ by $d(x) = (x - 3)$

(a) Does $x - 3$ divide evenly in $f(x)$?

(b) Calculate the value of $f(3)$.

$$f(3) = -18$$

$$\begin{array}{r} x^3 - 2x^2 - 4x - 4 \\ x-3 \overline{) x^4 - 5x^3 + 2x^2 + 8x - 6} \\ \underline{x^4 - 3x^3} \\ -2x^3 + 2x^2 + 8x - 6 \\ \underline{-2x^3 + 6x^2} \\ -4x^2 + 8x - 6 \\ \underline{-4x^2 + 12x} \\ -4x - 6 \\ \underline{-4x + 12} \\ -18 \end{array}$$

$r \neq 0$

The Remainder and Factor Theorem

Remainder Theorem:

If a polynomial $f(x)$ is divided by $x - a$, then the remainder is equal to $f(a)$.

Factor Theorem:

A polynomial $f(x)$ has a factor $(x - a)$ if and only if $f(a) = 0$.

The Rational Zero Test

What if we are not given any roots or zeros of the polynomial but we are still asked to factor it. For example, in Example 1 we were told to divide $f(x) = x^3 - 3x^2 + 4$ by $x - 2$ and in doing that we were able to completely factor out $f(x)$. Next, we are interested in how to find potential zeros. This is the result of The Rational zero Test.

The Rational Zero Test

$$f(x) = (x-2)(x-2)(x+1)$$

If the polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

has integer coefficients, then every rational zero of f has the form

$$\text{Rational zero} = \frac{p}{q} \quad \text{reduced form}$$

where p and q have no common factors other than 1, p is a factor of the constant term a_0 , and

q is a factor of the leading coefficient a_n .

$$\text{Possible rational zeros} = \frac{\text{factors of constant term } a_0}{\text{factors of leading coefficient } a_n}$$

Example 4: Find the rational zeros of

$$4 = 1 \cdot 4$$

$$4 = (-1)(-4)$$

$$4 = 2 \cdot 2$$

$$4 = (-2)(-2)$$

1. $f(x) = x^3 - 3x^2 + 4 = (x+1)(x-2)^2$

constant term = 4

factors of 4: $\pm 1, \pm 2, \pm 4$

leading coefficient = 1

factors of 1: ± 1

By Rational zero Test
possible rational zeros are:
 $\pm 1, \pm 2, \pm 4$

$$f(1) \neq 0$$

$$f(-1) = 0$$

$$f(2) = 0$$

$$f(4) \neq 0$$

$$f(-2) \neq 0$$

$$f(-4) \neq 0$$

2. $h(x) = 2x^3 + 3x^2 - 8x + 3$

factors of constant term 3: $\pm 1, \pm 3$

factors of leading coeff 2: $\pm 1, \pm 2$

possible rational zeros:

$$\pm 1, \pm 3, \pm 1/2, \pm 3/2$$

possible zeros

$$f(1) = 0$$

$$f(-1) =$$

$$f(1/2) = 0$$

$$f(-1/2) =$$

$$f(3) =$$

$$f(-3) = 0$$

$$f(3/2) =$$

$$f(-3/2) =$$

The zeros are $1, 1/2, -3$
 $h(x) = (x-1)(x+3)(x-1/2) = (x-1)(x+3)(2x-1)$

3. $g(x) = x^3 + x + 1$

lead term

const term

factors of const term: ± 1

factors of leading term: ± 1

possible rational zeros:

$$\frac{\pm 1}{\pm 1} = \pm 1$$

$$f(-1) = (-1)^3 + (-1) + 1 = -1$$

$$f(1) = 1 + 1 + 1 = 3$$

None of them are zeros
it has an irrational root (or zero)
between -1 and 0.

rational numbers: $1/2, -3/2, 1/3$

irrational numbers: $\pi, \sqrt{2}, e$

The Fundamental Theorem of Algebra

Real numbers: rational + irrational

An n th degree polynomial can have at most n real roots.

In the complex number system, every n th degree polynomial has exactly n zeros.

Recall Complex Numbers

$$\sqrt{-1} = i \text{ or } i^2 = -1$$

$$\sqrt{-15} = \sqrt{(-1)(15)} = \sqrt{-1} \sqrt{15} = i\sqrt{15} \rightarrow \text{complex number}$$

$$\sqrt{-9} = \sqrt{(-1)9} = \sqrt{-1} \cdot \sqrt{9} = i3 = 3i \rightarrow \text{complex number}$$

A complex number is a number of the form $a + bi$, where a and b are constants.

a is called the real part

b is called the imaginary part

EX $2 + 3i$ $a=2$, $b=3 \rightarrow$ complex number

EX $\sqrt{5}i = i\sqrt{5}$ $a=0$, $b=\sqrt{5} \rightarrow$ complex number / pure imaginary

EX $-17 = -17 + 0i$, $a=-17$, $b=0$
 $\Rightarrow$ real number

Note: -17 is a both a complex number and a real number.

Note Every real number is a complex number, but not every complex number is a real number.

Types of Zeros of a polynomial function f

• Real

rational: $f(x) = (2x+1)(x-1)$

rational zeros: $-1/2, 1$

Graphically
real zeros
are x -intercepts

irrational: $f(x) = (x-1)(x^2-2)$

$x=1, -\sqrt{2}, \sqrt{2}$

irrational
zeros

$$x^2 - 2 = 0$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

• Complex
zeros: $2i, 1-i, 1+3i$

If $1-i$ is a zero then $1+i$ is zero also.

Complex zeros come in pairs!

$1-i$ and $1+i$ are
conjugates

Fundamental Theorem of Algebra

Every polynomial function with degree $n \geq 1$ can be factored into n linear factors of the following form:

$$f(x) = a(x - r_1) \cdot (x - r_2) \cdot (x - r_3) \cdot \dots \cdot (x - r_n)$$

where $r_1, r_2, r_3, \dots, r_n$ are complex numbers and $a \neq 0$.

Example 5: Completely factor the polynomial over the complex numbers.

$$f(x) = (x - 1)(x^2 + 4)(x^2 - 3)$$

$$f(x) = (x - 1)(x^2 + 4)(x - \sqrt{3})(x + \sqrt{3})$$

gives me complex roots

factored completely over the real numbers

$$f(x) = (x - 1)(x - \sqrt{3})(x + \sqrt{3})(x - 2i)(x + 2i)$$

factored completely over the complex numbers

Example 6: List all possible rational zeros. Find all zeros and completely factor f over the complex numbers.

$$f(x) = 2x^4 + x^3 + 17x^2 + 9x - 9$$

*real: rational + irrat.
complex: come in pairs*

const. term: -9 factors: $\pm 1, \pm 3, \pm 9$

leading term: 2 factors: $\pm 1, \pm 2$

possible rational zeros: $\pm 1, \pm 1/2, \pm 3, \pm 3/2, \pm 9, \pm 9/2$

Actual rational zeros: $x = -1$ and $x = 1/2$ (from the graph)

$(x + 1)$ and $(x - 1/2) = (2x - 1)$

Multiply $(x + 1)(2x - 1) = 2x^2 - x + 2x - 1 = 2x^2 + x - 1$

*2x - 1 = 0
2x = 1
x = 1/2*

$$\begin{array}{r}
 x^2 + 9 \\
 2x^2 + x - 1 \overline{) 2x^4 + x^3 + 17x^2 + 9x - 9} \\
 \underline{2x^4 + x^3 - x^2} \\
 18x^2 + 9x - 9 \\
 \underline{18x^2 + 9x - 9} \\
 0
 \end{array}$$

$$2x^4 + x^3 + 17x^2 + 9x - 9 = (2x^2 + x - 1)(x^2 + 9)$$

$$= (x+1)(2x-1)(x-3i)(x+3i)$$

$$x^2 + 9 = 0$$

$$x^2 = -9$$

$$x = \pm \sqrt{-9}$$

$$x = \pm 3i$$

$$x = 3i$$

$$x = -3i$$