

1. Functions and Relations

In nearly every physical phenomenon, we observe that one quantity depends on another. For example, the **cost of tuition** depends on **the number of credits you take**, or the **area of a circle** depends on its **radius**.

We use the term **function** to describe this dependence of one quantity on another.

We will use letters such as $f, g, h \dots$ to represent functions.

We can represent a function in **four different ways**. For example, we can use the letter f to represent a **rule** that **doubles the input** and **then add 1**.

1. **Verbally:** f is the rule that doubles the input x and then adds 1.

2. **Algebraically:**

$$y = 2x + 1$$

$$f(x) = 2x + 1$$

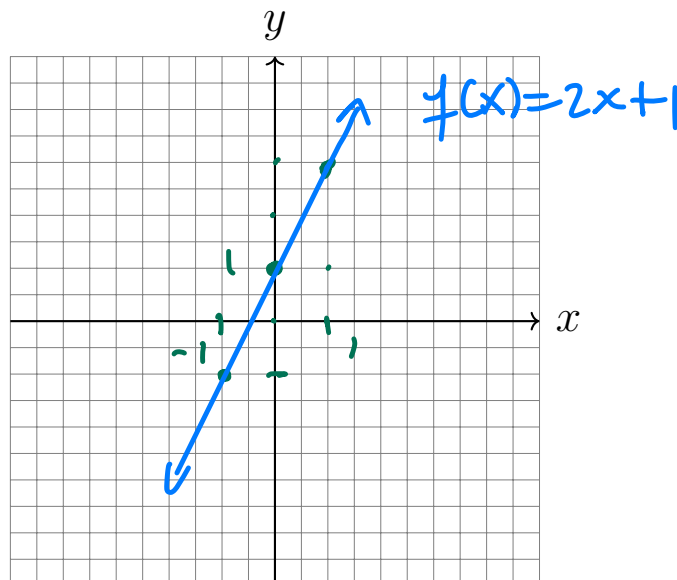
x : input or independent variable
 y : output or dependent variable

3. **Numerically:**

x	-1	0	1	2
y	-1	1	3	5

a table

4. **Graphically:**



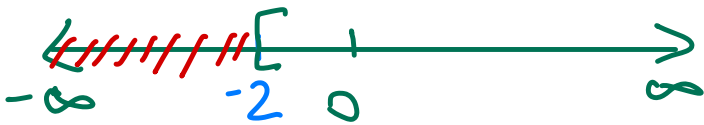
Formal definition of a function:

A function f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B .

Set A is called the domain of the function and it consists of all valid inputs.

Set B is called the range of the function and it consists of all possible outputs $f(x)$ as x varies throughout the domain.

Example 1. (Interval notation for Domains and Range) Find the domain and range for $f(x) = \sqrt{x+2}$



$$\begin{aligned}\sqrt{0} &= 0 \\ \sqrt{-1} &= \text{not real } \# \\ \sqrt{-2+2} &= \sqrt{0} = 0\end{aligned}$$

Domain: $[-2, \infty)$

Range: $[0, \infty)$

Example 2. Identify if the following relations are functions. If yes, identify the domain and range.

1. $y = x^2 + 1$

$\uparrow$ $\uparrow$
 $x = -1$
 $y = (-1)^2 + 1 = 2$

yes

Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

2.

x	y
1	2
1	3
2	4

> 1 has two output NO

3.

x	y
1	5
2	7
3	5

yes Domain: $\{1, 2, 3\}$
Range: $\{5, 7\}$

2. Piecewise defined functions

Sometimes a function has different rules for different portions of the domain and when that happens we call those **piecewise defined functions**.

$$f(x) = \begin{cases} \text{formula 1, piece 1 of domain} \\ \text{formula 2, piece 2} \\ \text{formula 3, piece 3} \end{cases}$$

$$f(x) = \begin{cases} 2x+1, & x \leq 0 \\ 3, & x > 0 \end{cases}$$

Example 3. So, for example, if a cell phone company charges \$40 a month if you use less than 500 minutes, but for every minute beyond the allowed 500 minutes you get charged a quarter a minute, the function that calculates the cost of the phone plan will be a piecewise defined function.

Find a formula for the cost of this phone plan, $C(x)$ where x is the number of minutes used in a month period.

$$C(x) = \begin{cases} 40, & x < 500 \end{cases}$$

$$\begin{cases} 40 + 0.25(x - 500), & x \geq 500 \end{cases}$$

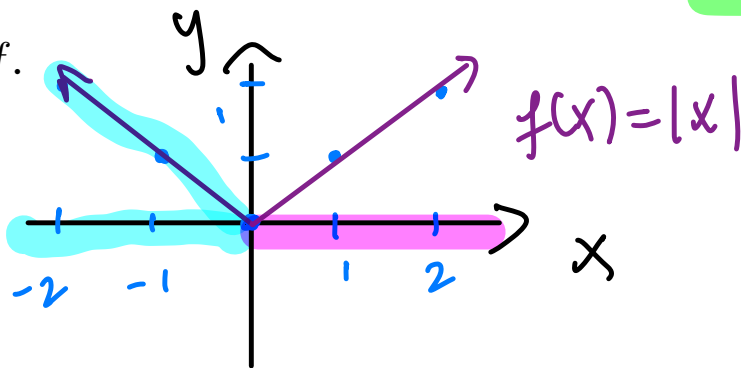
$$x = 510$$

$$40 + 10(0.25)$$

$$40 + 0.25(510 - 500) = 40 + 0.25(10) = 42.5$$

Example 4. Recall the absolute value function $f(x) = |x|$.

1. Graph f .



$$f(2) = |2| = 2$$

$$f(-3) = |-3| = 3$$

2. Rewrite f as a piecewise defined function.

$$f(x) = \begin{cases} -1x + 0, & x \leq 0 \\ +x + 0, & x > 0 \end{cases}$$

$$f(x) = \begin{cases} -x, & x \leq 0 \\ x, & x > 0 \end{cases}$$

3. Evaluating a Function

Evaluating a function at a specific input value means plug in that specific value in the formula to get your output.

Example 5. Let $f(x) = 2x^2 + 3x + 1$. Evaluate

1. $f(2) = 2(2)^2 + 3(2) + 1 = 15$

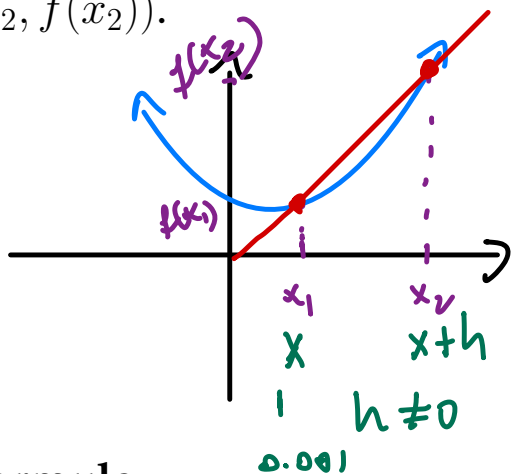
2. $f(-3) = 2(-3)^2 + 3(-3) + 1 = 2 \cdot 9 - 9 + 1 = 10$

3. $f(a) = 2a^2 + 3a + 1$

4. Difference Quotient

The difference quotient represents the **average rate of change** of a function between two values x and $x + h$.

Recall that the Average rate of change of a function f between two points x_1 and x_2 is just the slope of the line going through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.



Average rate of change = slope of secant line

$$\frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Formula:

$$\text{Difference quotient} = \frac{f(x+h) - f(x)}{(x+h) - x}$$

$$\text{Difference quotient} = \frac{f(x+h) - f(x)}{h}$$

⊗!

Example 6. Find the difference quotient for $f(x) = 2x^2 + 3x + 1$.

$$\begin{aligned}
 \frac{f(x+h) - f(x)}{h} &= \frac{2(x+h)^2 + 3(x+h) + 1 - (2x^2 + 3x + 1)}{h} \\
 &= \frac{2(x^2 + 2hx + h^2) + 3x + 3h + 1 - 2x^2 - 3x - 1}{h} \\
 &= \frac{\cancel{2x^2} + 4hx + 2h^2 + \cancel{3x} + 3h + \cancel{1} - \cancel{2x^2} - \cancel{3x} - \cancel{1}}{h} \\
 &= \frac{4hx + 2h^2 + 3h}{h} = \cancel{h}(4x + 2h + 3) \\
 &= \underline{4x + 3} + \cancel{2h}
 \end{aligned}$$

$(x+h)^2 = (x+h)(x+h)$
 $= x^2 + \underline{xh} + \underline{hx} + h^2$
 $(x+h)^2 \neq x^2 + h^2$

Goal is to cancel the h of the denominator

Example 7. Find the difference quotient for $f(x) = \frac{1}{x+1}$.

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{(x+1)(x+h+1)} - \frac{1}{(x+1)(x+h+1)}}{h}$$

$$\begin{aligned}
 &= \frac{\frac{x+1 - (x+h+1)}{(x+1)(x+h+1)}}{h} = \frac{\cancel{x+1} - \cancel{x} - h - \cancel{1}}{(x+1)(x+h+1)} \cdot \frac{h}{1} \\
 &= \frac{-h}{(x+1)(x+h+1)} = \frac{\cancel{h}}{(x+1)(x+h+1)} \cdot \frac{1}{\cancel{h}} \\
 &= \frac{1}{(x+1)(x+h+1)}
 \end{aligned}$$

Practice 1: Find the domain of functions:

1. $f(x) = \frac{\sqrt{2+x}}{3-x}$ $\rightarrow 2+x$ can't be negative
 $\rightarrow$ denominator can't be zero
 $3-x \neq 0$ so $x \neq 3$



Domain: $[-2, 3) \cup (3, \infty)$

Try these
on your
own!

2. $g(x) = \sqrt[3]{x-1}$

Quantities under (odd) roots can be negative. So domain is all real numbers

Domain $(-\infty, \infty)$

3. $h(x) = \frac{x^4}{(x-3)(x+3)}$ $\rightarrow$ denominator can't be zero.
 $x \neq 3, x \neq -3$



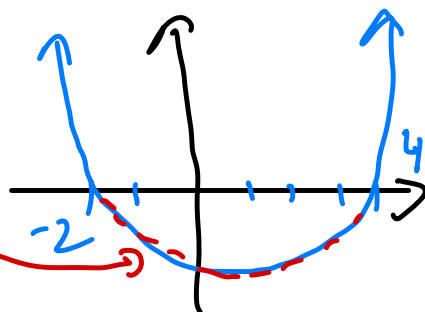
Domain $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

4. $f(t) = \sqrt{t^2 - 2t - 8}$

First factor the quantity under square root.

$t^2 - 2t - 8 = (t+2)(t-4)$

$(t+2)(t-4)$ can't be negative



Domain: $(-\infty, -2] \cup [4, \infty)$

Practice 2: Evaluate (if possible) the function $g(x) = \frac{1-x}{1+x}$ at:

$$1. g(2) = \frac{1-2}{1+2} = -\frac{1}{3}$$

$$3. g(a-1) = \frac{1-(a-1)}{1+a-1} = \frac{2-a}{a}$$

$$2. g(-1) = \frac{1-(-1)}{1+(-1)} = \frac{2}{0}$$

not possible

$$4. g(x^2-1) = \frac{1-(x^2-1)}{1+x^2-1} = \frac{2-x^2}{x^2}$$

Practice 3: Find the difference quotient for $f(x) = x^2 + 2x - 3$.

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^2 + 2(x+h) - 3 - (x^2 + 2x - 3)}{h} \\ &= \frac{\cancel{x^2} + 2xh + h^2 + \cancel{2x} + 2h - \cancel{3} - \cancel{x^2} - \cancel{2x} + \cancel{3}}{h} \\ &= \frac{2xh + h^2 + 2h}{h} = \frac{h(2x + h + 2)}{h} = 2x + 2 + h \end{aligned}$$

Practice 4: Evaluate the piecewise defined function at the indicated values:

$$f(x) = \begin{cases} x^2 + 2x & x \leq -1 \\ x & -1 < x \leq 1 \\ -1 & 1 < x \end{cases}$$

$$1. f(-4) = (-4)^2 + 2(-4) = 8$$

$$3. f(-1) = (-1)^2 + 2(-1) = 1 - 2 = -1$$

$$2. f(25) = -1$$

$$4. f(0) = 0$$

5. Self Assessment

Take a moment to reflect on your understanding of each learning objective. For each topic, write the appropriate number from 1 to 5, where 1 star means you don't understand the concept at all, and 5 means you feel completely confident about it. Be honest with your assessment—this will help you identify areas where you might need additional practice or support.

1. **Determine if a relation is a function:**
2. **Use function notation and evaluate functions:**
3. **Find the domain of functions:**
4. **Evaluate difference quotients:**