

1. Adding, subtracting, multiplying and dividing functions

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg , and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers.

ALGEBRA OF FUNCTIONS

Let f and g be functions with domains A and B . Then the functions $f + g$, $f - g$, fg , and f/g are defined as follows.

$$(f + g)(x) = f(x) + g(x) \quad \text{Domain } A \cap B$$

$$(f - g)(x) = f(x) - g(x) \quad \text{Domain } A \cap B$$

$$(fg)(x) = f(x)g(x) \quad \text{Domain } A \cap B$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \text{Domain } \{x \in A \cap B \mid g(x) \neq 0\}$$

Practice: Perform the following function operations for f and g and find the domains of the resulting functions.

$$f(x) = \frac{1}{x-2} \text{ and } g(x) = \sqrt{x}.$$

(a) $f + g$

(b) $f - g$

(c) fg

(d) $\frac{f}{g}$

2. Composition of Functions

Another way to combine two functions f and g is to compose them. Composing the function $f(x)$ with the function $g(x)$, denoted $(f \circ g)(x)$, means that the output of g becomes the input of f . In other words $(f \circ g)(x) = f(g(x))$ or the function f is evaluated at $g(x)$.

Example: Let $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{4}{x-1}$. Find the following:

(a) $f \circ g$ and its domain

(b) $g \circ f$ and its domain

NOTE The domain of $f \circ g$ is the intersection of the domain of inner function g and the resulting function $f \circ g$.

Practice: Exercise 1

Let $f(x) = \frac{x}{x+3}$ and $g(x) = 8x - 3$. Find the following

1. $(f \circ g)(x)$ and its domain

2. $(g \circ f)(x)$ and its domain

Practice: Exercise 2

Let $f(x) = x^3 + 8$ and $g(x) = x - 5$ and $h(x) = \sqrt{x}$. Find $f \circ g \circ h$.

Practice: Exercise 3

A spherical balloon is being inflated. The radius of the balloon is increasing at a rate of 3cm/s.

1. Find a function f that models the radius as a function of time t , in seconds.

2. Find a function g that models the volume as a function of the radius r , in cm.

3. Find and interpret $g \circ f$.

6. One-to-one Functions

A function f is **one-to-one** if there is a one-to-one correspondence between the inputs and the outputs. In other words, f never takes on the same values twice.

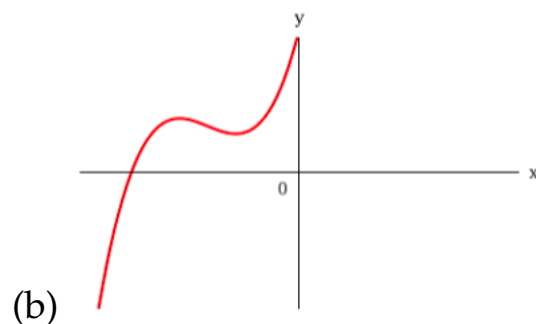
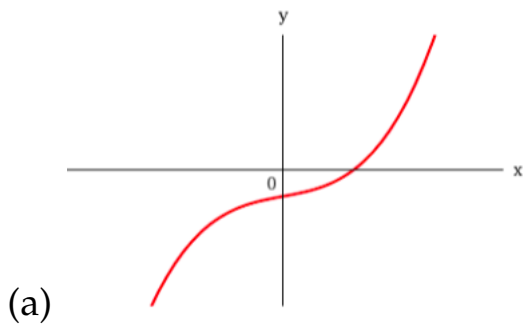
Examples: Is f one-to-one?

$$f(x) = x^2$$

$$f(x) = x^3$$

How to determine if a function f is one-to-one - Graphically

Determine if the functions f and g are one-to-one.



Practice - Restricted Domain

Graph the function $f(x) = (x + 2)^2 - 1$ and decide if it is one-to-one on its entire domain.

7. Inverse Functions

A one-to-one function with domain A and range B has an **inverse function** which takes on as domain the range of f and gives as output the domain of f . In other words the inverse of a function f undoes what f does. The inverse, denoted as f^{-1} takes as input the output of f and gives as output the input of f , i.e. if $f(x) = y$ then $f^{-1}(y) = x$.

For example if f is the function that takes the input x multiplies it by 5, adds 2 then takes the 5th power of the result, what would the inverse f^{-1} have to be to undo what f did?

The inverse function f^{-1} reverses the effect of f !

INVERSE FUNCTION PROPERTY

Let f be a one-to-one function with domain A and range B . The inverse function f^{-1} satisfies the following cancellation properties:

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

Conversely, any function f^{-1} satisfying these equations is the inverse of f .

Practice: Are f and g inverses of each other?

$$f(x) = 8 - 4x, \quad g(x) = \frac{8 - x}{4}$$

NOTE: The function and its inverse switch domain and range:

Domain of $f =$

Range of $f =$

8. How to find the inverse of a one-to-one function - Algebraically

Let f be a one-to one function. Then $f(x) = y$ for any x in the domain of f and f has an inverse denoted f^{-1} .

Example: Let $f(x) = \frac{x - 9}{x + 9}$.

(a) Is f one-to-one?

(b) Find the domain of f .

(c) Can you find the range of f without graphing it?

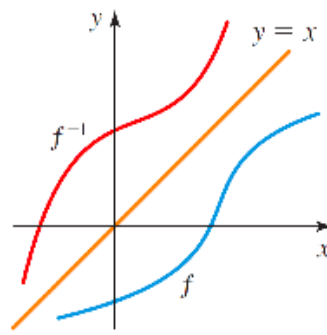
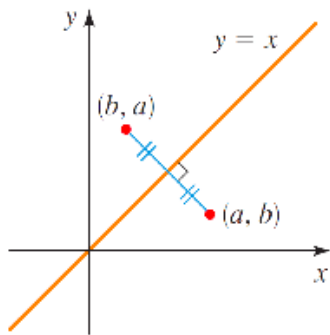
(d) Find the inverse of f .

(e) Can you find the range of f now using the information you have about f^{-1} ?

9. How to find the inverse of a one-to-one function - Graphically

Let f be a one-to-one function. Then $f(x) = y$ for any x in the domain of f and f has an inverse f^{-1} . A point (a, b) on the graph of f is going to become (b, a) on the graph of f^{-1} since the input is switched with the output.

The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.



Example: Given the graph of f below find the graph of f^{-1} .

