Day 13: MTH 122
5.1 Quadratic Functions and Applications

1 Quadratic Functions and Applications

1.1 Definition and Forms of Quadratic Functions

Definition: Quadratic Function

A **quadratic function** is a function of the form:

$$f(x) = ax^2 + bx + c$$

where a, b, and c are real numbers, and $a \neq 0$.
The graph of a quadratic function is called a **parabola**.

Definition: Forms of Quadratic Functions

A quadratic function can be written in two main forms:
General Form:

$$f(x) = ax^2 + bx + c$$

Vertex Form:

$$f(x) = a(x-h)^2 + k$$

where (h, k) is the vertex of the parabola.

Definition: Properties of Quadratic Functions

For a quadratic function $f(x) = a(x-h)^2 + k$ in vertex form:

1. **Opens upward** if $a > 0$, **opens downward** if $a < 0$.
2. The **vertex** is at the point (h, k).
3. The graph has a **minimum** value of k if $a > 0$.
4. The graph has a **maximum** value of k if $a < 0$.
5. The **axis of symmetry** is the vertical line $x = h$.
6. The **domain** is all real numbers (i.e., $(-\infty, \infty)$).
7. The **range** is $[k, \infty)$ if $a > 0$, or $(-\infty, k]$ if $a < 0$.

Example 1: Analyzing a Quadratic Function in Vertex Form

Analyze the quadratic function $f(x) = -2(x-1)^2 + 8$ by finding:
a) Whether the parabola opens upward or downward
b) The vertex
c) The axis of symmetry
d) The maximum or minimum value
e) The y-intercept
f) The x-intercepts
g) The domain and range
h) A sketch of the graph

Solution:

Example 2: Converting from General Form to Vertex Form

For the quadratic function $f(x) = 3x^2 + 12x + 5$:

a) Convert to vertex form by completing the square

b) Identify the vertex

c) Determine whether the parabola opens upward or downward

d) Find the axis of symmetry

e) Find the maximum or minimum value

f) Find the y-intercept

g) Find the x-intercepts (if any)

h) Determine the domain and range

i) Sketch the graph

Solution:

Definition: Vertex Formula

For a quadratic function $f(x) = ax^2 + bx + c$, the x-coordinate of the vertex is:

$$x = -\frac{b}{2a}$$

The y-coordinate can be found by evaluating $f\left(-\frac{b}{2a}\right)$ or by using:

$$y = c - \frac{b^2}{4a}$$

Therefore, the vertex is at $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$.

Example 3: Using the Vertex Formula

Find the vertex of the quadratic function $f(x) = -2x^2 + 12x - 13$.

Example 4: Converting to Vertex Form Using the Vertex Formula

Convert the quadratic function $f(x) = 4x^2 - 16x + 25$ to vertex form using the vertex formula.

1.2 The Discriminant and Number of Solutions

Definition: The Discriminant

For a quadratic equation $ax^2 + bx + c = 0$, the **discriminant** is defined as:

$$\Delta = b^2 - 4ac$$

The discriminant determines the number of real solutions to the equation:

- If $\Delta > 0$, then the equation has
- If $\Delta = 0$, then the equation has
- If $\Delta < 0$, then the equation has

Note: Graphical Interpretation of the Discriminant

Graphically, for a parabola $y = ax^2 + bx + c$:

- If $\Delta > 0$, the parabola intersects the x-axis at two distinct points.
- If $\Delta = 0$, the parabola is tangent to the x-axis (touches it at exactly one point).
- If $\Delta < 0$, the parabola doesn't intersect the x-axis (it lies entirely above or below the x-axis).

Example 5: Using the Discriminant

For each of the following quadratic equations, calculate the discriminant and determine the number of real solutions:

a) $2x^2 - 3x + 1 = 0$
b) $4x^2 + 12x + 9 = 0$
c) $3x^2 - 2x + 5 = 0$

1.3 Applications of Quadratic Functions

Example 6: Projectile Motion

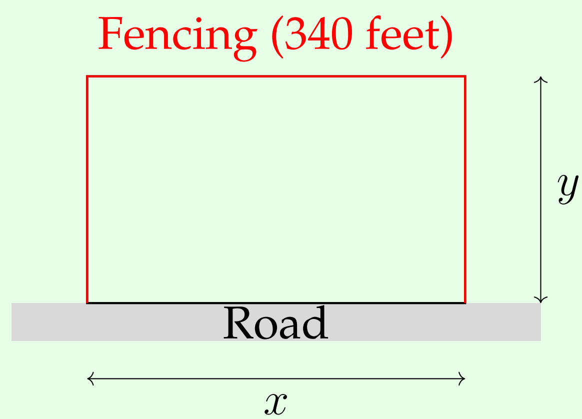
A ball is thrown upward from a height of 6 feet with an initial velocity of 64 feet per second. The height $h(t)$ of the ball (in feet) after t seconds is given by:

$$h(t) = -16t^2 + 64t + 6$$

a) Find the maximum height reached by the ball.
b) At what time does the ball reach its maximum height?
c) When does the ball hit the ground?
d) Sketch the graph of the height function.

Example 7: Maximizing Area

A parking lot is to be constructed next to a road with 340 feet of fencing available for the remaining three sides. What are the dimensions of the parking lot that would yield the maximum area?



Solution:

2 Summary of Key Concepts

Key Concepts Summary

1. A **quadratic function** has the form $f(x) = ax^2 + bx + c$ where $a \neq 0$.
2. The **vertex form** of a quadratic function is $f(x) = a(x-h)^2 + k$ where (h, k) is the vertex.
3. A parabola **opens upward** if $a > 0$ and **opens downward** if $a < 0$.
4. The **axis of symmetry** is the vertical line passing through the vertex: $x = h$ or $x = -\frac{b}{2a}$.
5. If $a > 0$, the function has a **minimum** value at the vertex. If $a < 0$, it has a **maximum** value.
6. The **vertex formula**: For $f(x) = ax^2 + bx + c$, the vertex is at $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$.
7. The **discriminant** $\Delta = b^2 - 4ac$ determines the number of real solutions:
   - $\Delta > 0$: Two distinct real solutions
   - $\Delta = 0$: One real solution (repeated root)
   - $\Delta < 0$: No real solutions
8. **Completing the square** is a method to convert a quadratic function from general form to vertex form.
9. Applications of quadratic functions include:
   - Projectile motion
   - Maximizing area or revenue
   - Minimizing cost or time