# Day 6, Monday Feb 10th

- Exam 1 on Wednesday
- Written HW 1 due today by 11:59 pm
- Practice Problems for exam 1 posted in WebAssign
  (material on exam - everything but
  arithmetic sequences)

**Example 7.** Write the terms for the sum and evaluate the sum.
$\sum_{n=1}^{5} 2n+3$ $= 5 + (2\cdot 2+3) + (2\cdot 3+3) + (2\cdot 4+3) + (2\cdot 5+3)$
$= 5 + 7 + 9 + 11 + 13 = 45$

**Note:** In Calculus you will learn how to find what an infinite series adds up to for certain types of series.

**Example 8.** Consider the series:

$a = 1/2$, $r = 1/2$ $\rightarrow S = \frac{1/2}{1-1/2} = \frac{\frac{1}{2}}{\frac{1}{2}}$ $S=1$

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

convergent series to 1

Lets investigate the first few partial sums:

$S_1 =$ $1/2 = 0.5$

$S_2 =$ $1/2 + 1/4 = 3/4 = 0.75$

$S_3 =$ $S_2 + a_3 = 3/4 + 1/8 = 0.875$

$S_4 =$ $0.9375$

$S_5 = 0.96875$

$\vdots$

As we continue adding terms, the sum gets closer and closer to 1.

**Example 9.** Consider the series:

$$\sum_{i=1}^{\infty} \frac{1}{i} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

divergent!

Lets investigate the first few partial sums:

$S_1 =$ $1$

$S_2 =$ $1 + 1/2 = 1.5$

$S_3 =$ $1.8333$

$S_4 =$ $2.0833\ldots$

As we continue adding terms, the sum grows larger and larger without bound. This series does not add up to an actual value.

## 4. Factorial notation

Factorials form the basis for important series like Taylor series expansions.

**Definition.** The factorial of a positive integer $n$, written as $n!$, is the product of all positive integers. less than or equal to $n$.

For example, $5! = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120$ $\quad 5! = 5\cdot 4\cdot 3\cdot 2\cdot 1$

$10! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10 = 3628800$

$n! = 1\cdot 2\cdot 3\cdot 4\cdot \cdots \cdot (n-2)\cdot(n-1)\cdot n$

By definition, $0! = 1$

**Example 10.** Evaluate the following expressions.

Write the bigger factorial in terms of smaller factorial.

(a) $\dfrac{8!}{3!\cdot 5!}$

$$\frac{\cancel{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot \cancel{6}\cdot 7\cdot 8}{\cancel{3!}\ \cancel{5!}} = 56$$

$8! = 5!\ 6\cdot 7\cdot 8$

(b) $\dfrac{n!}{(n+2)!} = \dfrac{\cancel{n!}}{\cancel{n!}\,(n+1)(n+2)} = \dfrac{1}{(n+1)(n+2)}$

$(n+2)! = \underbrace{1\cdot 2\cdot 3\cdot \cdots \cdot n}\,(n+1)\,(n+2) = n!\,(n+1)(n+2)$

(c) $\dfrac{(n+1)!}{(n-1)!} = \dfrac{\cancel{1\cdot 2\cdot 3\cdots(n-1)}\,n\,(n+1)}{\cancel{(n-1)!}}$

$\dfrac{n+1}{n-1}$ , 1 , $n(n+1)$ , $n!$

(d) $\dfrac{(2n)!}{(n)!} = \dfrac{\cancel{1\cdot 2\cdot 3\cdots n}\,(2n-3)(2n-2)(2n-1)(2n)}{\cancel{n!}}$

$= (n+1)(n+2)(n+3)\cdots(2n)$

**Example 11.** Given the sequence defined by $b_n = \dfrac{n^2}{(n+1)!}$ find $b_1$ and $b_6$.

$$b_1 = \frac{1^2}{(1+1)!} = \frac{1}{2!} = \frac{1}{2}$$

$$b_6 = \frac{6^2}{(6+1)!} = \frac{36}{7!} = \frac{\cancel{36}}{\cancel{1\cdot 2\cdot 3}\cdot 4\cdot 5\cdot \cancel{6}\cdot 7} = \frac{1}{140}$$

**Day 6 notes**
**Feb 10th**
*8.2/8.3 Arithmetic and Geometric Sequences*

## 1. Arithmetic Sequences

In this section we are going to introduce Arithmetic Sequences. The simplest way to generate an arithmetic sequence is to start with a number $a$ and add to it a fixed constant $d$, over and over again.

can be negative

**Definition:** An arithmetic sequence is a sequence of the form:

$a, a+d, a+2d, a+3d, \ldots$

$a_1 \quad a_2 \quad a_3 \quad a_4$

The number $a$ is the first term and $d$ is called the common difference.

The nth term of an arithmetic sequence is given by:

$$\boxed{a_n = a + (n-1)d}$$

$a_2 - a_1 = d$

$a_4 - a_3 = d$

$a_1 = a$

$a_2 = a + 1d$

$a_3 = a + 2d$

**Example 1**: Is this sequence an arithmetic sequence? $13, 7, 1, -5, \cdots$.

$-6 \quad -6 \quad -6$

$7 - 13 = -6 \qquad -5 - 1 = -6$

$1 - 7 = -6$

If yes, find the common difference, the next three terms, the nth term, and the 300th term of the arithmetic sequence

$d = -6$

$a = 13$

$a_5 = -5 - 6 = -11$

$a_6 = -11 - 6 = -17$

$a_7 = -17 - 6 = -23$

$\boxed{a_n = a + (n-1)d}$ $\qquad$ $\boxed{a_n = 13 + (n-1)(-6)}$

$a_{300} = 13 + (300-1)(-6) = -1781$ ✓

**Example 3:** Write the first five terms of the sequence $a_n = -4 + 3n$. Determine whether or not the sequence is arithmetic. If it is, find the common difference.

$a_1 = -4+3 = -1$
$a_2 = 2$
$a_3 = 5$
$a_4 = 8$
$a_5 = 11$

+3 +3 +3 +3

If yes

$\boxed{d = 3}$

$\boxed{a = -1}$

**Example 2:** The 11th term of an arithmetic sequence is 52, and the 19th term is 92. Find the 1000th term.

$a_{11} = 52$

$a_{19} = 92$

$a_{1000} = ?$

$52 = a + 10d$
$92 = a + 18d$

subtract

$40 = (\cancel{a} + 18d) - (\cancel{a} + 10d)$

$40 = 8d$

$d = 5$

$52 = a + 10(5)$

$52 = a + 50$

$a = 2$

$a_{1000} = 4997$

$a_n = a + (n-1)d$

$a_{11} = a + (11-1)d$

$a_{11} = a + 10d$

$a_{19} = a + 18d$

$a_{1000} = 2 + (999)5 = 4997$

**Arithmetic sequence recursive formula:**

$a_1 = a$

$a_n = a_{n-1} + d$

$$\boxed{\begin{aligned} a_1 &= a \\ a_n &= a + (n-1)d \end{aligned}}$$

$a_n = 2a_{n-1} + 5$

$a_n =$

Recall:

Sequence : $a_1, a_2, a_3, \ldots$

Series : $a_1 + a_2 + a_3 + \ldots = \sum_{i=1}^{\infty} a_i$

Partial sum : $a_1 + a_2 + \cdots + a_{10} = S_{10} = \sum_{k=1}^{10} a_k$

Factorial : $n! = 1 \cdot 2 \cdot 3 \cdots n$

Arithmetic : $a_n = a + (n-1)d$

$a$: first term

$d$: common difference

$a_2 - a_1$ or $a_5 - a_4 = d$

## 2. Partial Sums of Arithmetic Sequences

**PARTIAL SUMS OF AN ARITHMETIC SEQUENCE**

$a_1, a_2, a_3, \ldots$ $\;+d \;\; +d$

For the arithmetic sequence given by $a_n = a + (n-1)d$, the ***n*th partial sum**

$$S_n = \underbrace{a}_{a_1} + \underbrace{(a+d)}_{a_2} + \underbrace{(a+2d)}_{a_3} + \underbrace{(a+3d)}_{a_4} + \cdots + \underbrace{[a+(n-1)d]}_{a_n}$$

is given by either of the following formulas.

**1.** $S_n = \dfrac{n}{2}[2a + (n-1)d]$ **2.** $S_n = n\left(\dfrac{a + a_n}{2}\right)$ ✱

**Example 4:** a) Find the sum of the first 100 numbers.

$1+2+3+4+\cdots+100 = ?$

$\{1,2,3,4,\ldots\}$ is arithmetic w/ $a=1$, $d=1$

✱ $S_{100} = 100\left(\dfrac{1+100}{2}\right) = 100 \cdot \dfrac{101}{2} = 5050$

b) Find the sum of the first 50 odd numbers.

$1+3+5+7 \ldots\ldots \overset{99}{a_{50}} = ?$

$S_{50} = ?$ $S_n = n\left(\dfrac{a+a_n}{2}\right)$

$a = 1$

$d = 2$

$a_{50} = 1 + (50-1)2 = 99$

$a_n = a + (n-1)d$

$S_{50} = 50\left(\dfrac{1+99}{2}\right) = 2500$

$a = 3$, $d = -4$

c) Sum the first 25 terms of the sequence $3, -1, -5, -9, \cdots -93$

$S_{25} = ?$ $-1125$

$a_n = a + (n-1)d$

$a_{25} = 3 + (25-1)(-4) = -93$

$S_{25} = 25\left(\dfrac{3+(-93)}{2}\right)$

$S_{25} = -1125$

$S_n = n\left(\dfrac{a+a_n}{2}\right)$

Next, we are going to introduce Geometric Sequences. A geometric sequence is generated when we start with a number $a$ and repeatedly multiply by a fixed nonzero constant $r$.

**Definition:** A geometric sequence is a sequence of the form:

$a, ar, ar^2, ar^3, \cdots, ar^{n-1}$

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_n$

The number $a$ is the first term and $r$ is called the common ratio.

$r = \frac{a_2}{a_1}, \frac{a_3}{a_2} = r, \frac{a_{100}}{a_{99}} = r$

The nth term of a geometric sequence is given by:

$$\boxed{a_n = ar^{n-1}}$$

**Example 5**: If $a = 3$ and $r = 2$, then we have the geometric sequence:

$3, 6, 12, 24, \ldots$

$a_{10} = ?$

$a_{10} = 3(2)^{10-1}$
$= 1536$

**Example 6**: The sequence
$2, –10, 50, –250, 1250, \cdots$ is a geometric sequence with $a = 2$ and $r = –5$. Find $a_{10}$.

$a_{10} = ar^{10-1} = 2(-5)^9 = -3{,}906{,}250$

**Example ~~7~~**: Find the common ratio, the first term, the nth term, and the eighth term of the geometric sequence $5, 15, 45, 135, \cdots$

$a$ $\times 3$ $\times 3$

$r = 3$

$a_8 = 5(3)^{8-1} = 10{,}935.$

## 4. Partial Sums of Geometric Sequences

The partial sum $s_n$ of a geometric sequence is given by the formula below.

**PARTIAL SUMS OF A GEOMETRIC SEQUENCE**

For the geometric sequence defined by $\boxed{a_n = ar^{n-1}}$, the ***n*th partial sum**

$$S_n = \underset{a_1}{a} + \underset{a_2}{ar} + \underset{a_3}{ar^2} + \underset{a_4}{ar^3} + \underset{a_5}{ar^4} + \underset{\cdots}{\cdots} + \underset{a_n}{ar^{n-1}} \qquad r \neq 1$$

is given by

$$S_n = a\frac{1-r^n}{1-r}$$

**Example 7:** Find the following partial sum of a geometric sequence:
$1 + 4 + 16 + \cdots + 4096$

$a_1 + a_2 + a_3 + \cdot \cdot + a_n = a_7$

yes geom w/ $a_1 = 1$, $r = 4$

$S_n = a \dfrac{1-r^n}{1-r} \Rightarrow S_7 = 1 \dfrac{1-4^7}{1-4} = (1-4^7) \div (-3) = 5461$

Need to find $n$: use $a_n = ar^{n-1}$

$4096 = 1 \cdot 4^{n-1}$

$\Downarrow$

$4096 = 4^{n-1}$

$6 = n-1 \Rightarrow n = 7 \qquad 4^6 = 4^{n-1}$

## 5. Infinite Geometric Series

An infinite geometric series is a series of the form
$a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + \cdots$.

**SUM OF AN INFINITE GEOMETRIC SERIES**

If $|r| < 1$, then the infinite geometric series

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \cdots$$

converges and has the sum

$$S = \frac{a}{1-r}$$

If $|r| \geq 1$, the series diverges.

**Example 8:** Determine whether the infinite geometric series is convergent or divergent. If it is convergent, find its sum.

(a) $2 + \frac{2}{5} + \frac{2}{25} + \frac{2}{125} + \cdots$ $= 2.5$

(b) $1 + \frac{7}{5} + \frac{49}{25} + \cdots$

b) $r = 7/5$ $\quad \left|\frac{7}{5}\right| \geq 1$ so divergest

$a = 1$ $\quad 1.4 \geq 1$

a) $a = 2$, $r = \frac{1}{5}$

Since $|r| = \left|\frac{1}{5}\right| = \frac{1}{5} < 1$ the infinite series converges to $S = \frac{a}{1-r} = \frac{2}{1-\frac{1}{5}}$

$$S = \frac{2}{\frac{5}{5}-\frac{1}{5}} = \frac{2}{\frac{4}{5}} = \cancel{2} \times \frac{5}{\cancel{4}_2} = \frac{5}{2} = 2.5$$