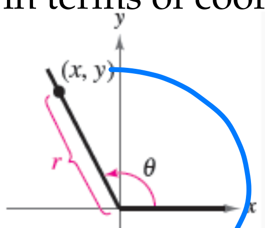
In our previous work with trigonometric functions, we were restricted to acute angles in right triangles. Now, we will extend these concepts to find trigonometric functions for any angle. This allows us to apply trigonometry to a much wider range of problems and to develop a more complete understanding of these important functions. We will see how trigonometric functions can be defined for angles in standard position, how reference angles help us find values of trigonometric functions, and how we can evaluate trigonometric functions for any real number input.

4.4.1 Trigonometric Functions for Angles in Standard Position

When we place an angle θ in standard position, we can define the trigonometric functions in terms of coordinates on the terminal side.



Definition Trigonometric Functions for Any Angle

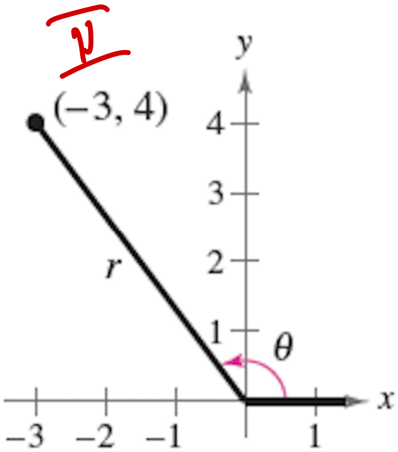
Let θ be an angle in standard position, and let (x, y) be any point (except the origin) on the terminal side of θ. If $r = \sqrt{x^2 + y^2}$ is the distance from the origin to the point (x, y), then:

$$\boxed{\begin{aligned} \sin\theta &= \frac{y}{r} \\ \cos\theta &= \frac{x}{r} \end{aligned}}$$

$$\begin{aligned}
\tan\theta &= \frac{y}{x} = \frac{\sin\theta}{\cos\theta} \quad (x \neq 0) \\
\csc\theta &= \frac{r}{y} = \frac{1}{\sin\theta} \quad (y \neq 0) \\
\sec\theta &= \frac{r}{x} = \frac{1}{\cos\theta} \quad (x \neq 0) \\
\cot\theta &= \frac{x}{y} = \frac{\cos\theta}{\sin\theta} \quad (y \neq 0)
\end{aligned}$$

Example 1

Example 1. Let $(-3, 4)$ be a point on the terminal side of θ. Find the sine, cosine, and tangent of θ.



$\sin\theta = \frac{y}{r} = \frac{4}{5}$ $\csc\theta = \frac{5}{4}$

$\cos\theta = \frac{x}{r} = -\frac{3}{5}$ $\sec\theta = \frac{5}{-3}$

$\tan\theta = \frac{y}{x} = \frac{4}{-3}$ $\cot\theta = \frac{-3}{4}$

Solution:

$(x, y) = (-3, 4)$

$x = -3$

$y = 4$

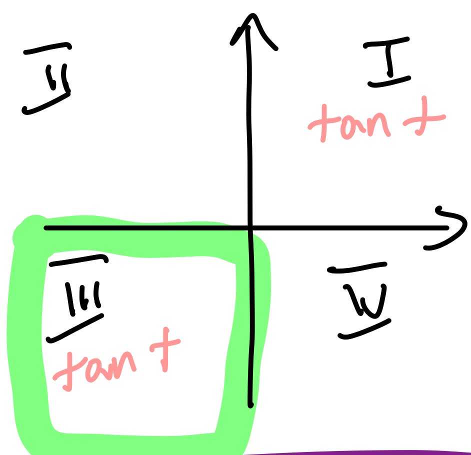
Find $r = \sqrt{x^2+y^2} = \sqrt{(-3)^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$

Example 2

Example 2. Given $\sin\theta = -\frac{2}{3}$ and $\tan\theta > 0$, find $\cos\theta$ and $\cot\theta$.

Solution:

- θ is in Q III



$\sin\theta = -\frac{2}{3}$, use $\sin\theta = \frac{y}{r}$ $\Big\}$ $y = -2$, $r = 3$

To find $\cos\theta$:

$\cos\theta = \frac{x}{r} = \frac{x}{3} = \frac{-\sqrt{5}}{3}$

need x so use $r = \sqrt{x^2+y^2}$

$3 = \sqrt{x^2+4}$ 2

$9 = x^2+4$

$x^2 = 5$

$x = \pm\sqrt{5}$

$x = -\sqrt{5}$

$\cot\theta = ?$

$\cot\theta = \frac{x}{y} = \frac{-\sqrt{5}}{-2} = \frac{\sqrt{5}}{2}$

4.4.2 Reference Angles

For any angle in standard position, we can define a reference angle that helps us find trigonometric values.

theta prime

Definition Reference Angle

The reference angle θ' for an angle θ is the acute angle formed by the terminal side of θ and the x-axis.

Note

For an angle θ in standard position:

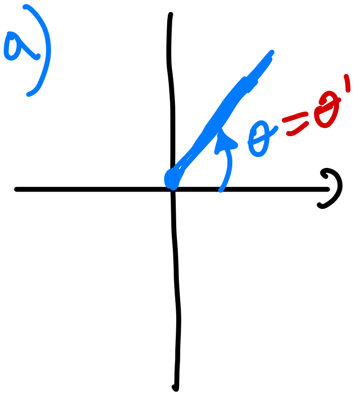
a) If θ is in the first quadrant ($0°$ to $90°$), then $\theta' = \theta$

b) If θ is in the second quadrant ($90°$ to $180°$), then $\theta' = 180° - \theta$

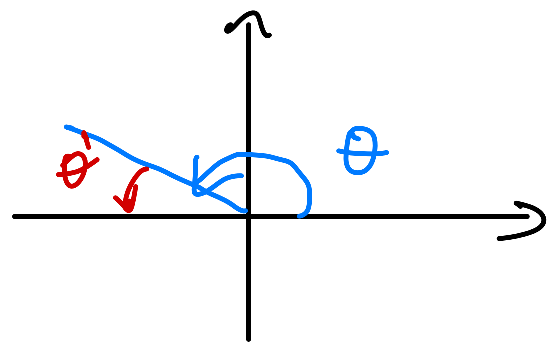
c) If θ is in the third quadrant ($180°$ to $270°$), then $\theta' = \theta - 180°$

d) If θ is in the fourth quadrant ($270°$ to $360°$), then $\theta' = 360° - \theta$

a)



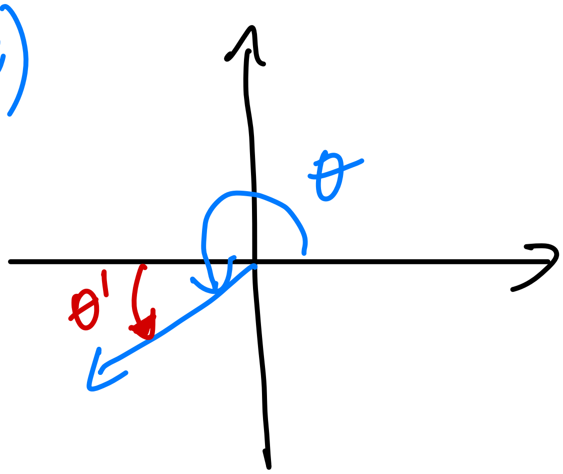
b)



$\theta' = 180° - \theta$

$\theta' = \pi - \theta$

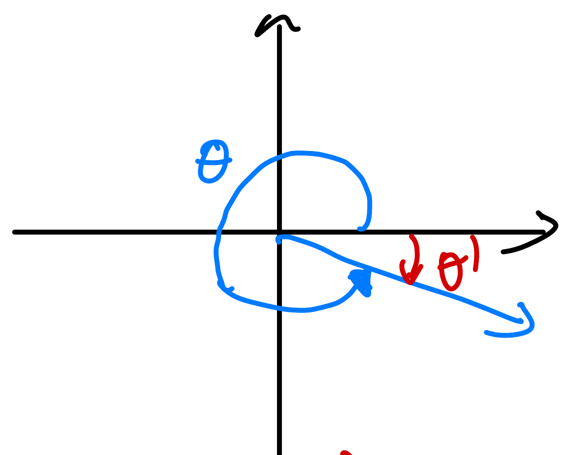
c)



$\theta' = \theta - 180°$

$\theta' = \theta - \pi$

d)



$\theta' = 360° - \theta$

$\theta' = 2\pi - \theta$

Example 3

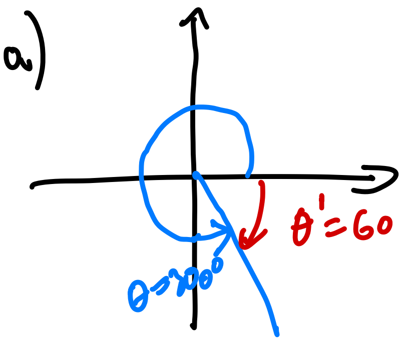
Example 3. Find the reference angle θ' for each value of θ.

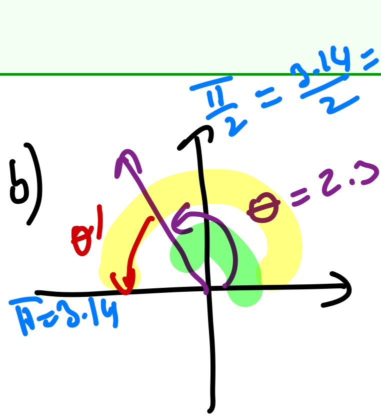
a. $\theta = 300°$

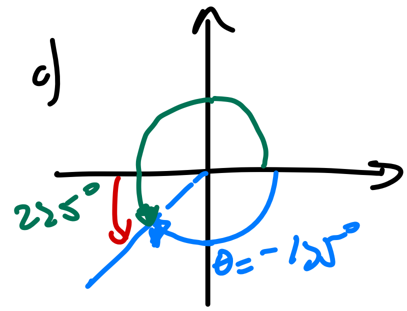
b. $\theta = 2.3$

c. $\theta = -135°$

Solution:







$\theta' = 360° - 300$
$= 60°$

$\theta' = \pi - 2.3$
$\theta' \approx 0.84$

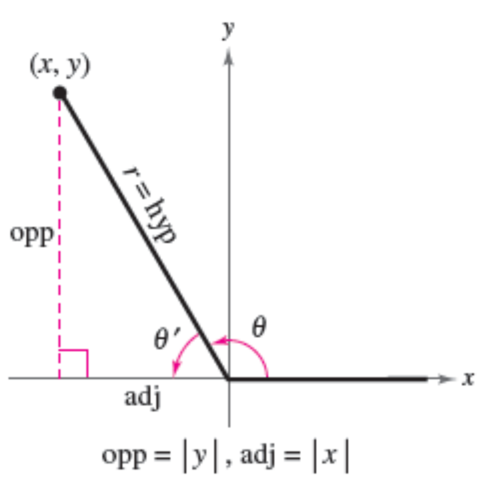
need a coterminal positive to $-135°$
Ans: $225°$
$\theta' = 45°$
$\theta' = \theta - 180° = 225° - 180°$
$= 45°$

4.4.3 Trigonometric Functions of Real Numbers

We can extend the domain of trigonometric functions from angles to any real number by interpreting the real number as the radian measure of an angle.

Definition Trigonometric Functions of Real Numbers

If θ is a real number, then $\sin\theta$, $\cos\theta$, $\tan\theta$, etc., are defined as the trigonometric values of the angle whose radian measure is θ.



Values of $\sin\theta$, $\cos\theta$, $\tan\theta$, etc are going to be the same values as $\sin\theta'$, $\cos\theta'$, $\tan\theta'$, etc except possibly in sign.

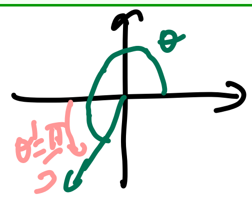
Example 4

Example 4. Evaluate each trigonometric function.
a. $\cos\frac{4\pi}{3}$ b. $\tan(-210°)$ c. $\csc\frac{11\pi}{4}$

$f(-x) = -f(x)$

a) $\theta = \frac{4\pi}{3}$

$\theta' = ?$

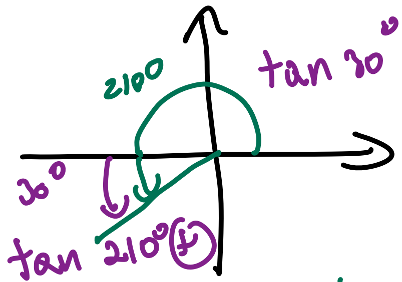


Find $\boxed{\cos\frac{\pi}{3} = \frac{1}{2}}$

$\cos\frac{4\pi}{3} = -\frac{1}{2}$

b) $\tan(-210°) = -\underbrace{\tan 210°}$

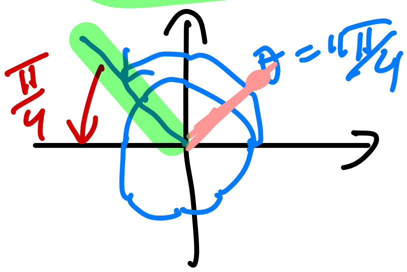
$\tan 30°$ is (+) $= -\frac{\sqrt{3}}{3}$



$\tan 30° = \frac{\sin 30°}{\cos 30°} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

c) $\boxed{\csc\frac{11\pi}{4}} = \frac{1}{\sin\frac{11\pi}{4}} = \frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2}$ $\sin\frac{11\pi}{4} = ?$ $\frac{\sqrt{2}}{2}$

$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$



Example 5

Example 5. Let θ be an angle in Quadrant II such that $\sin\theta = \frac{1}{3}$. Find $\cos\theta$ by using trigonometric identities.

Solution:

$\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta + \left(\frac{1}{3}\right)^2 = 1$

$\cos^2\theta = 1 - \frac{1}{9} = \frac{9}{9} - \frac{1}{9} = \frac{8}{9}$

$\cos^2\theta = \frac{8}{9}$

$\cos\theta = \pm\sqrt{\frac{8}{9}} \Rightarrow \boxed{\cos\theta = -\frac{2\sqrt{2}}{3}}$

(!!!)

Sine and Cosine Functions

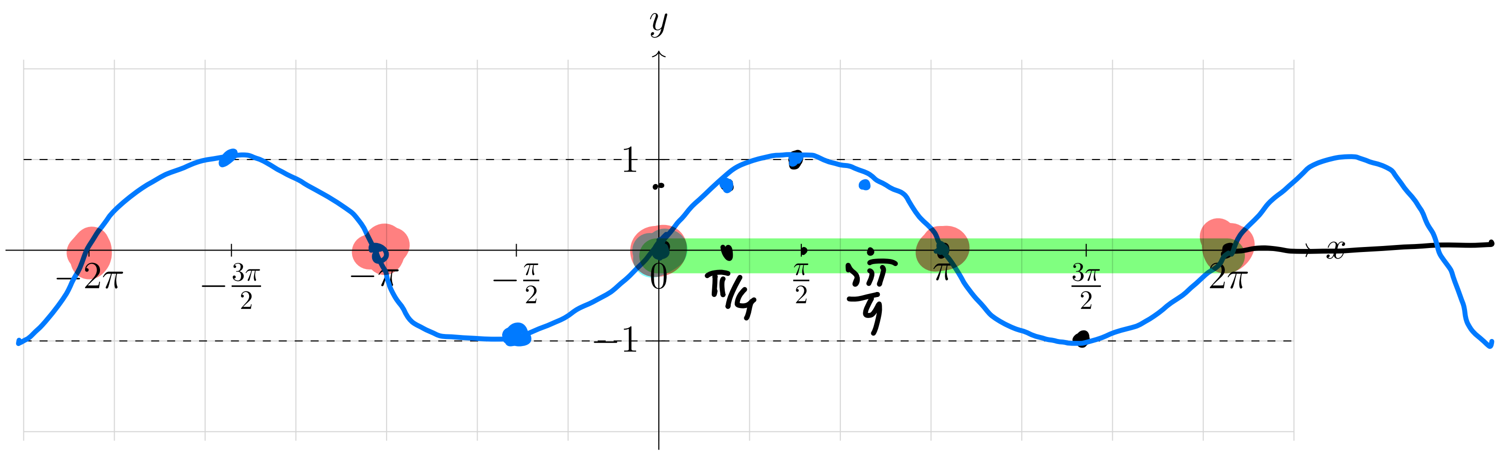
Definition Sine and Cosine Functions

$$y = \sin x$$
$$y = \cos x$$

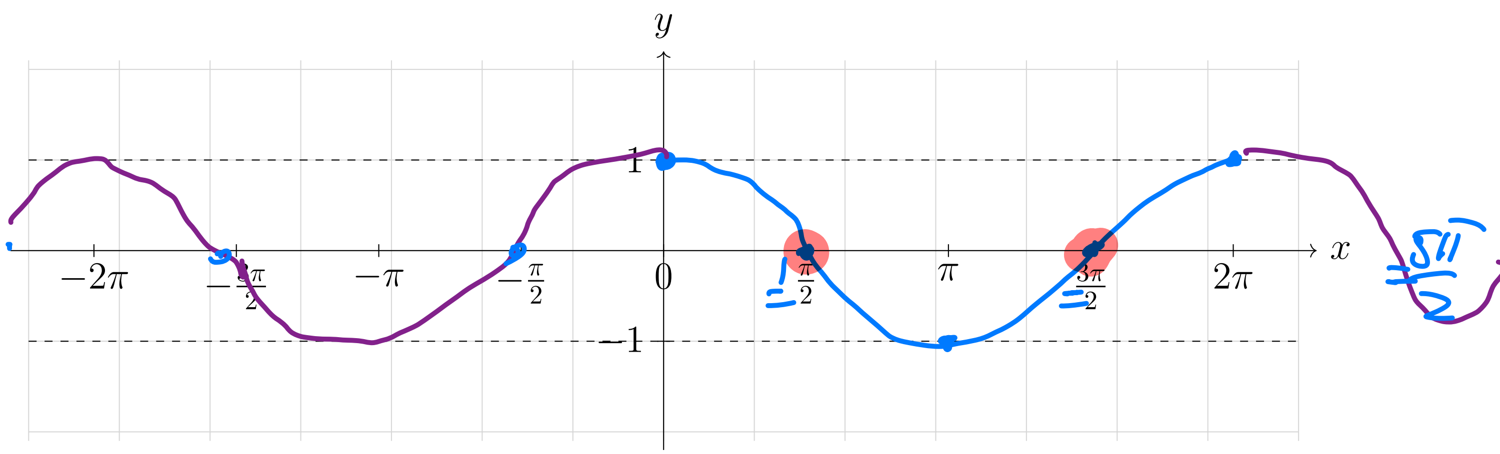
$f(x) = \sin x$
$y = \sin x$

where x is in radians.

Graph of Sine Function



Graph of Cosine Function



Key Features of Sine and Cosine

Properties of Sine Function	Properties of Cosine Function
1) domain: $(-\infty, \infty)$	1) domain: $(-\infty, \infty)$
2) range: $[-1, 1]$	2) range: $[-1, 1]$
3) symmetry: origin sym. odd $\sin(-x) = -\sin x$	3) symmetry: even $\cos(-x) = \cos x$ y axis symmetry
4) period: 2π	4) period: 2π
5) x-intercepts: $0, \pm\pi, \pm 2\pi, \ldots$ $n\pi$, n is integer	5) x-intercepts: odd multiples of $\frac{\pi}{2}$ $\pi + n\pi/2$, n is integer
6) y-intercept: $(0,0)$	6) y-intercept: $(0,1)$

Table 1: Properties of Sine and Cosine Functions

Transformations of Sine and Cosine

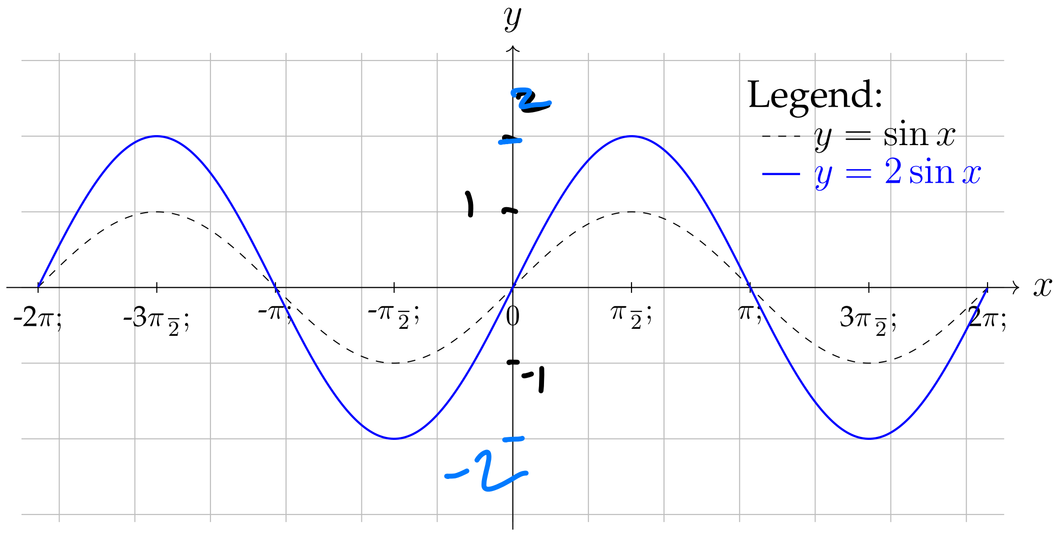
Definition General Form

$$y = a\sin(bx - c) + d$$
$$y = a\cos(bx - c) + d$$

Amplitude of sine and cosine functions

The constant factor a acts as a scaling factor- a vertical stretch or vertical srink of the basic sine curve.

- $|a| > 1$ vertical stretch
- $|a| < 1$ vertical shrink
- $|a|$ is called the amplitude of the function $y = a\sin(x)$



$a = 2$

$|a| = |2| = 2$

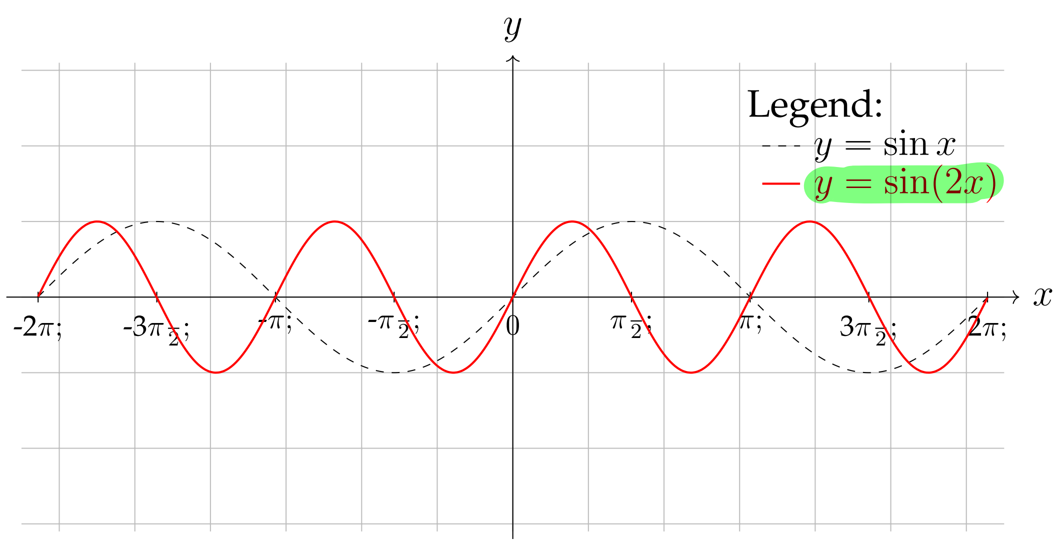
amplitude

Period of sine and cosine

Let b be a positive real number. The period of $y = a\sin(bx)$ and $y = a\cos(bx)$ is given by

$$\text{period} = \frac{2\pi}{b}$$

.



$b = 2$

$\text{period} = \frac{2\pi}{b}$

$= \frac{2\pi}{2} = \pi$

Translations of sine and cosine

Vertical translations: