3. Geometric Sequences

In this section we are going to introduce Geometric Sequences. A geometric sequence is generated when we start with a number a and repeatedly multiply by a fixed nonzero constant r.

DEFINITION OF A GEOMETRIC SEQUENCE

A **geometric sequence** is a sequence of the form

$$a, ar, ar^2, ar^3, ar^4, \ldots$$

The number a is the **first term**, and r is the **common ratio** of the sequence. The ***n*th term** of a geometric sequence is given by

$$a_n = ar^{n-1}$$

$a_1 = a$
$a_2 = ar$
$a_3 = arr = ar^2$
$a_4 = ar^3$
$\boxed{a_n = ar^{n-1}}$
$\frac{a_4}{a_3} = \frac{\cancel{a}r^3}{\cancel{a}r^2} = \boxed{r}$

Example 4: If $a = 3$ and $r = 2$, then we have the geometric sequence:

$a_1 = a = 3$
$a_2 = ar = 3 \cdot 2 = 6$
$a_3 = ar^2 = 3 \cdot 2^2 = 12$
$a_4 = 3 \cdot 2^3 = 24$

$\{6, 12, 24, 48, \ldots\}$

$\boxed{a_n = ar^{n-1} = 3 \cdot 2^{n-1}}$

Example 5:The sequence
$2, -10, 50, -250, 1250, \cdots$ is a geometric sequence with $a = 2$ and $r = -5$.

$a_1 = a = 2$

$r = \frac{a_2}{a_1} = \frac{-10}{2} = -5$ or $\boxed{r = \frac{a_3}{a_2} = \frac{50}{-10} = -5}$

$\boxed{a_{10} = ar^{10-1} = 2(-5)^9}$

Example 6: Find the common ratio, the first term, the nth term, and the eighth term of the geometric sequence $5, 15, 45, 135, \cdots$

$\boxed{r = \frac{15}{5} = 3}$ $\boxed{a = 5}$

$a_n = ar^{n-1}$

$\boxed{a_n = 5(3)^{n-1}}$

$\boxed{a_8 = 5(3)^{8-1} = 5 \cdot 3^7}$

$\boxed{a_8 = 10{,}935}$

4. Partial Sums of Geometric Sequences

The partial sum s_n of a geometric sequence is given by the formula below.

PARTIAL SUMS OF A GEOMETRIC SEQUENCE

For the geometric sequence defined by $a_n = ar^{n-1}$, the **nth partial sum**

$a, ar, ar^2, ar^3, \ldots$

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} \qquad r \neq 1$$

is given by

$$\boxed{S_n = a\frac{1-r^n}{1-r}} \quad *$$

Example 6: Find the following partial sum of a geometric sequence:
$1 + 4 + 16 + \cdots + 4096$

$\{1, 4, 16, \ldots, 4096\}$

$S_n = a\frac{1-r^n}{1-r}$

$\boxed{a=1}$

$\boxed{r=\frac{4}{1}=4}$

$a_n = 4096$

$ar^{n-1} = 4096$

$4^{n-1} = 4096$

$4^{n-1} = 4^6$

$n-1=6$

$n=7$

$S_7 = 1\frac{1-4^7}{1-4} = 5461$

5. Infinite Geometric Series

An infinite geometric series is a series of the form
$\underset{a_1}{a} + \underset{a_2}{ar} + \underset{a_3}{ar^2} + ar^3 + ar^4 + \cdots + \underset{a_n}{ar^{n-1}} + \cdots$. $= S$

SUM OF AN INFINITE GEOMETRIC SERIES

If $|r| < 1$, then the infinite geometric series

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$$

converges and has the sum

$$S = \frac{a}{1-r}.$$

If $|r| \geq 1$, the series diverges.

Example 7: Determine whether the infinite geometric series is convergent or divergent. If it is convergent, find its sum.

(a) $\boxed{2 + \frac{2}{5} + \frac{2}{25} + \frac{2}{125} + \cdots}$

(b) $1 + \frac{7}{5} + \frac{49}{25} + \cdots$

geom sequence $\{2, \frac{2}{5}, \frac{2}{25}, \frac{2}{125}, \cdots\}$

geom series $2 + \frac{2}{5} + \frac{2}{25} + \frac{2}{125} + \cdots = S = \frac{a}{1-r}$

(a) Find $r = ?$

$a_1 = a = 2$

$r = \frac{2/5}{2} = \frac{2}{5} \cdot \frac{1}{2} = \boxed{\frac{1}{5}}$

$|r| = |\frac{1}{5}| < 1 \Rightarrow$ geom series conv to

$\frac{a}{1-r} = \frac{2}{1-\frac{1}{5}} = \frac{2}{\frac{4}{5}} = 2 \cdot \frac{5}{4} = \boxed{\frac{5}{2}}$

$= \boxed{2.5}$

b) $r = \frac{7/5}{1} = \boxed{\frac{7}{5}}$

$|r| = |7/5| > 1$ so the geom series diverges