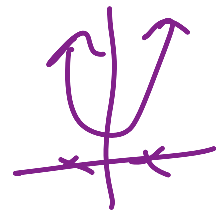
Day 15: MTH 122
1.3 Complex numbers

0.1 The Imaginary Number i

Definition 1. *The **imaginary number** i is defined as $i = \sqrt{-1}$, or equivalently, $i^2 = -1$.*

$i^2 = (\cancel{\sqrt{}}{-1})^{\cancel{2}} = -1$

This definition allows us to express the square roots of negative numbers. If $a > 0$, then:



$$\sqrt{-a} = \sqrt{(-1)a} = \sqrt{-1}\sqrt{a} = i\sqrt{a}$$

Example 1. *Express the following in terms of i:*

$$\sqrt{-9} = \sqrt{(-1)9} = \sqrt{-1}\sqrt{9} = i3 = 3i$$
$$\sqrt{-16} = 4i$$
$$\sqrt{-7} = \sqrt{-1(7)} = \sqrt{-1}\sqrt{7} = i\sqrt{7}$$

$\sqrt{7i}$?

0.2 Simplifying Expressions with i

When manipulating expressions involving i, remember that $i^2 = -1$.

Example 2. *Simplify $\sqrt{-5} \cdot \sqrt{-3}$:*

$$\begin{aligned} \sqrt{-5} \cdot \sqrt{-3} &= i\sqrt{5}\, i\sqrt{3} \\ &= i^2\sqrt{15} \\ &= -1 \cdot \sqrt{15} \\ &= -\sqrt{15} \end{aligned}$$

$\sqrt{ab} = \sqrt{a}\sqrt{b}$ only hold for $a, b > 0$

$\sqrt{5}\,\sqrt{3} = \sqrt{5 \cdot 3} = \sqrt{15}$

$\sqrt{-5}\,\sqrt{-3} \cancel{=} \sqrt{(-5)(-3)} = \sqrt{15}$

Example 3. *Simplify* $\frac{\sqrt{-8}}{\sqrt{-2}}$:

$$\frac{\sqrt{-8}}{\sqrt{-2}} = \frac{i\sqrt{8}}{i\sqrt{2}}$$
$$= 1\,\frac{\sqrt{8}}{\sqrt{2}}$$
$$= \frac{\sqrt{8}}{\sqrt{2}}$$
$$=$$
$$= \sqrt{\frac{8}{2}}$$
$$= \sqrt{4}$$
$$= 2$$
$$=$$

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$
$a, b > 0$

0.3 Complex Numbers

Definition 2. *A **complex number** is any number that can be written in the form $a + bi$, where a and b are real numbers and i is the imaginary unit.*

- *a is called the **real part***
- *b is called the **imaginary part***
- *If $b = 0$, then $z = a$ is a real number*
- *If $a = 0$ and $b \neq 0$, then $z = bi$ is called a **pure imaginary number***

Ex: $2+3i$ complex number

Ex: $2 + 0 \cdot i = 2$

$0+3i = 3i$

Definition 3. *The **complex conjugate** of $a + bi$ is* $a - bi$.

ex: $2+3i$ its conjugate is $2-3i$.
$1-i$ its conjugate is $1+i$.
$0+3i$ its conjugate is $0-3i = -3i$

0.4 Operations with Complex Numbers

0.4.1 Addition and Subtraction

To add or subtract complex numbers, combine their real and imaginary parts separately:

Example 4.

$$\begin{aligned}(3+2i)+(4-5i) &= (3+4)+(2-5)i \\ &= 7-3i\end{aligned}$$

$$\begin{aligned}(6-4i)-(2+3i) &= (6-2)+(-4-3)i \\ &= 4-7i\end{aligned}$$

0.4.2 Multiplication

To multiply complex numbers, use the distributive property and remember that $i^2=-1$:

Example 5.

$$\begin{aligned}(2+3i)(4-i) &= 8-2i+12i+(3i)(-i) \\ &= 8-2i+12i-3i^2 \\ &= 8+10i-3(-1) \\ &= 8+10i+3 \\ &= 11+10i\end{aligned}$$

$$\begin{aligned}(1+i)(1-i) &= 1-\cancel{i}+\cancel{i}-i^2 \\ &= 1-(-1) \\ &= 1+1 \\ &= 2\end{aligned}$$

$$\begin{aligned}(1+i)(1+i) &= 1+i+i+i^2 \\ &= 1+2i-1 \\ &= 2i\end{aligned}$$

0.5 Powers of i

Powers of i follow a cyclic pattern of 4:

$$\begin{aligned} i^1 &= i \\ i^2 &= -1 \\ i^3 &= i^1(i^2) = i(-1) = -i \\ i^4 &= i^2 \cdot i^2 = (-1)(-1) = 1 \end{aligned}$$

$$\begin{aligned} i^5 &= i \cdot i^4 = i \cdot 1 = i \\ i^6 &= i^2 \cdot i^4 = (-1)1 = -1 \\ i^7 &= i^3 i^4 = (-i)(1) = -i \\ i^8 &= i^4 \cdot i^4 = 1 \cdot 1 = 1 \end{aligned}$$

For higher powers, we can use the fact that $i^4 = 1$ to simplify

$$\boxed{i^n = i^r}$$

where r is the remainder when n is divided by 4.

17 ÷ 4 has reminder of 1

Example 6. *Find i^{17}:*

$17 = 16 + 1$

$i^{17} = i^{16} \cdot i = (i^4)^4 \cdot i = 1 \cdot i = i$

Example 7. *Find i^{35}:* $= ?$

35 ÷ 4 has a reminder of 3

$35 = 4 \cdot 8 + 3$

$i^{35} = i^3 = -i$