$f(x) = mx + b$
$f(x) = ax^2 + bx + c$

1. Section 3.2 Intro to Polynomial Functions

In this section we introduce polynomial functions, their properties and graphs.

POLYNOMIAL FUNCTIONS

A **polynomial function of degree n** is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and $a_n \neq 0$.

The numbers $a_0, a_1, a_2, \ldots, a_n$ are called the **coefficients** of the polynomial.

The number a_0 is the **constant coefficient** or **constant term**.

The number a_n, the coefficient of the highest power, is the **leading coefficient**, and the term $a_n x^n$ is the **leading term**.

Examples: $f(x) = \frac{1}{2}x^3 + 2x^2 + 5$ yes : degree $= 3$

$g(x) = 3x^5 + 5x^2 + 5x^{-1} + 3$ NO

Example:
Answer the following questions about the function

$$f(x) = -3x^4 + 2x^3 + x + 1$$

1. Is f a polynomial function? yes degree 4

2. What are the coefficients of the polynomial?

   $-3, 2, 1, 1$

3. What is the constant term?

   1

4. What is the leading coefficient?

   -3

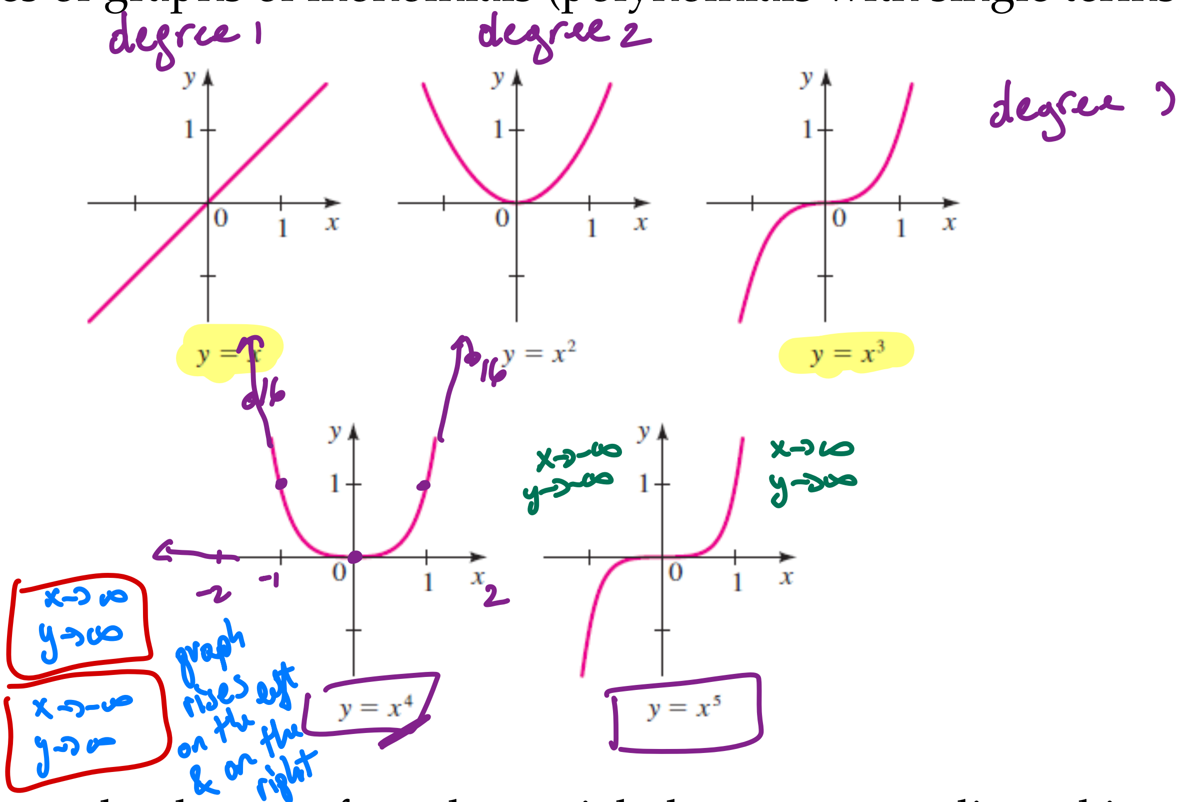
5. What is the leading term?

   $-3x^4$

2. Graphs of Polynomial Functions

Graphs of polynomial functions are smooth curves with no breaks or corners. (continuous functions)

Examples of graphs of monomials (polynomials with single terms):



The greater the degree of a polynomial, the more complicated its graph can be. The domain of a polynomial function is the set of all real numbers, so we can sketch only a small portion of the graph. In order to draw a sketch of a polynomial function we must take into consideration it intercepts (x intercepts are called zeroes or roots of the polynomial), its end behaviour and its turning points.

3. End behaviour of Polynomial Functions

The end behavior of a polynomial is a description of what happens as x becomes large in the positive or negative direction. The end behaviour of a polynomial is given by the leading term. To describe end behavior, we use the following arrow notation.

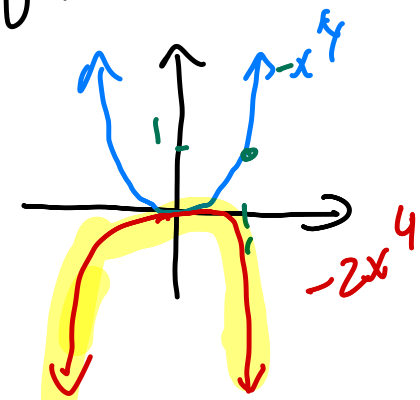
$x \to \infty$
$f(x) \to ?$

$x \to -\infty$
$f(x) \to ?$

$-f(x)$ or $f(-x)$

Example: Determine the end behaviour of the polynomial $P_1(x) = -2x^4 + 3x + 1$ and $P_2(x) = 5x^5 - 3x^4 + 3x^2 - 10.$

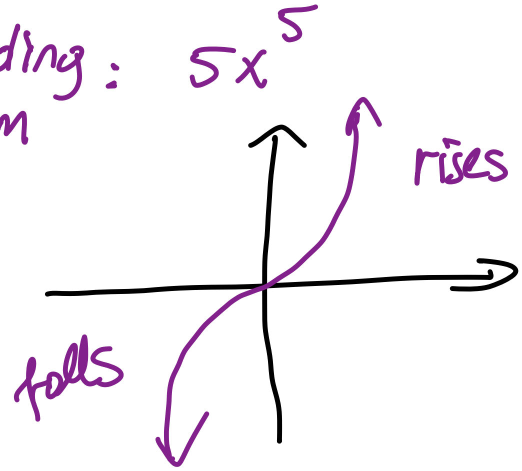
leading term: $-2x^4$



$x \to \infty$ $y \to -\infty$

$x \to -\infty$ $y \to -\infty$

leading term: $5x^5$



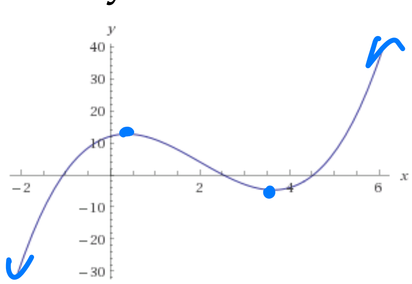
$x \to \infty$ $y \to \infty$

$x \to -\infty$ $y \to -\infty$

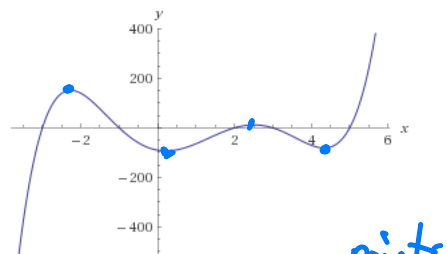
4. Turning points of Polynomial Functions

A turning point occurs whenever the graph of a polynomial function changes from increasing to decreasing or from decreasing to increasing. Turning points are associated with local maxima or local minima on a graph.

It is possible to use the graph to get a lower estimate of the degree of a polynomial. To do this, we just need to count the number of **turning points** (sometimes called "critical points") of the graph. A turning point is the same thing as a local max or min: it's a point where the graph switches between increasing and decreasing, or vice versa. For example, the polynomials below have 2 and 4 turning points, respectively:



2 turning points



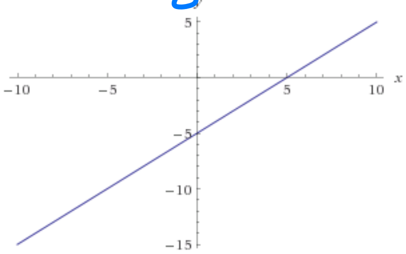
4 turning points

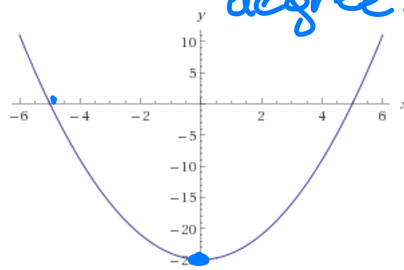
The number of turning points tells us the **smallest possible degree** the polynomial can have. The polynomial on the left must have degree at least $2+1=3$; the one on the right must have degree at least $4+1=5$. In general, if the graph of a polynomial has n turning points, the polynomial's degree is at least $n+1$.

Often, this formula will give the exact degree. For example, these polynomials

degree 1

degree 2



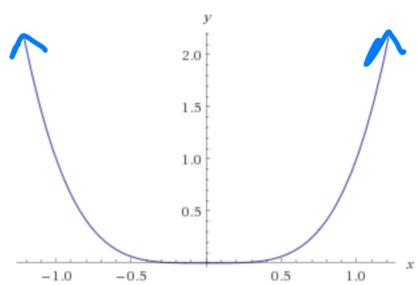


have 0 and 1 turning points, so their degrees are at least $0+1=1$ and $1+1=2$. But of course, these are just a linear function and quadratic function, so we know that the degrees are exactly 1 and 2.

It's not always exact, though. For example, this polynomial has 1 turning point

one turning point

degree at least 2



so its degree is at least $1+1=2$. But in fact, its degree is 4, so the estimate was not exact.

To find the zeros of a polynomial P, we factor and then use the Zero-Product Property.

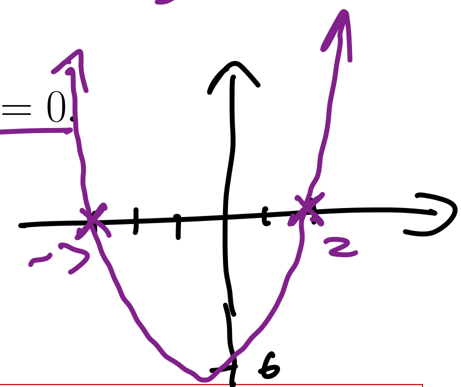
For example, to find the zeros of $P(x) = x^2 + x - 6$, we factor P to get $\boxed{P(x) = (x-2)(x+3)}$. From this factored form we easily see that → x intercepts : 2 & −3

(a) 2 is a zero of P.

(b) $x = 2$ is a solution of the equation $x^2 + x - 6 = 0$.

(c) $(x - 2)$ is a factor of $x^2 + x - 6$.

(d) 2 is an x-intercept of the graph of P.



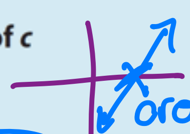
Multiplicity of zeroes Sometimes when we factor a polynomial there are factors that appear more than once. For example the polynomial $P(x) = (x+1)^2(x-2)(x-1)^3$ has the factor $(x+1)$ appearing twice. We say that its corresponding zero, $x = -1$, has multiplicity 2. Similarly the zero $x = 1$ has multiplicity 3. The shapes of the graphs around zeroes with even or add multiplicity looks different.

SHAPE OF THE GRAPH NEAR A ZERO OF MULTIPLICITY m

If c is a zero of P of multiplicity m, then the shape of the graph of P near c is as follows.

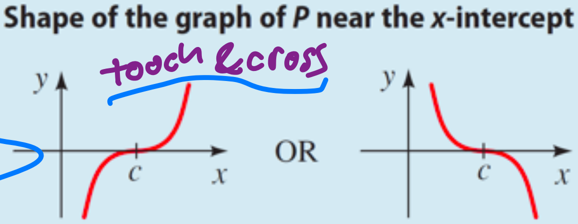
Multiplicity of c — **Shape of the graph of P near the x-intercept c**

m=1



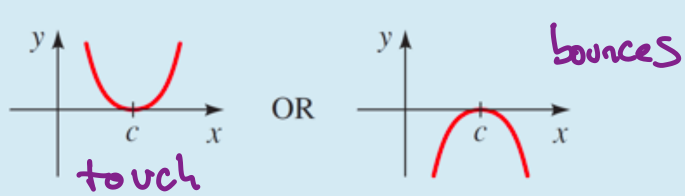
m odd, $m > 1$

3, 5, 7, ...



m even, $m > 1$

2, 4, 6, ...



degree : $x^2 \cdot x^3 \cdot x^1 = x^{2+3+1} = x^6$

Practice: Exercise 1

Graph the function $f(x) = -(x+1)^2(x-1)^3(x-2)$.

$x=-1$, $m=2$ (touch)

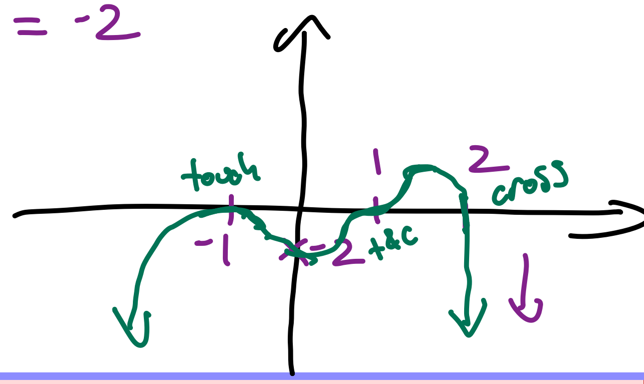
$x=1$, $m=3$ (touch & cross)

$x=2$, $m=1$ (cross)

y intercept: $f(0) = -(1)^2(-1)^3(-2) = -2$

leading term: $-x^6$

$x \to \infty$, $y \to -\infty$ $x \to -\infty$, $y \to -\infty$



Practice: Exercise 2

Factor the polynomial and use the factored form to find the real zeroes and then graph it $f(x) = x^3 + 5x^2 - 4x - 20$.

$f(x) = (x^3 + 5x^2) - 4x - 20$ factor by grouping

$f(x) = x^2(x+5) - 4(x+5)$

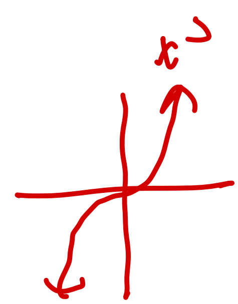
$= (x+5)(x^2-4)$

$a^2 - b^2 = (a-b)(a+b)$

$f(x) = (x+5)(x-2)(x+2)$

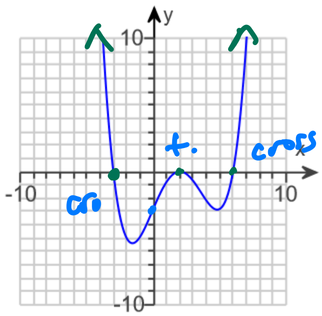
- x intercepts (zeros or roots of polynomial)
  $x=-5$ $m=1$, $x=2$ $m=1$, $x=-2$ $m=1$
- y intercepts : $(0,-20)$
- end behaviour
  $x \to \infty$, $y \to \infty$ $x \to -\infty$, $y \to -\infty$

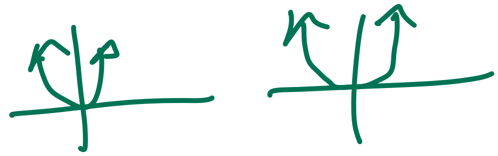
leading term : x^3



Practice Exercise 3

Use the graph of the polynomial function f to answer the following.





1. The x-intercepts are: $(-3,0), (2,0), (6,0)$
2. Is the leading coefficient positive or negative?
3. What is the minimum degree of f?
   3 turning points so min degree is 4.
4. Write a formula for $f(x) = a(x+3)(x-2)^2(x-6)$

3.3 Division of Polynomials and Factor and Remainder Theorem

Next, we are interested in how to get the factored form of higher degree polynomials so that we can get the real zeros fast. We will review, long division to divide polynomials. Then we will use that to factor higher degree polynomial and interpret the remainder (if any).

Example 1: Divide the polynomial $f(x) = x^3 - 3x^2 + 4$ by $x - 2$ and use the result to factor the polynomial completely.

dividend — divisor

$x^2 - x - 2$ ← result is the quotient

divisor: $x - 2$

$$\begin{array}{r} x^3 - 3x^2 + 0x + 4 \quad \rightarrow \text{dividend} \\ x^3 - 2x^2 \qquad\qquad \\ \hline 0 \quad -x^2 + 0x + 4 \quad \text{new dividend} \\ -x^2 + 2x \qquad \\ \hline 0 \quad -2x + 4 \quad \text{new dividend} \\ -2x + 4 \\ \hline 0 \rightarrow \text{remainder} \end{array}$$

② $\dfrac{x^3}{x} = x^2$

③ $(x-2)x^2$

④ $-3x^2 - (-2x^2)$
$= -3x^2 + 2x^2$
$= -x^2$

② $\dfrac{-x^2}{x} = -x$

$f(x) = (x-2)(x^2-x-2) = (x-2)(x-2)(x+1) = (x-2)^2(x+1)$

zeros: $x = 2,\ m = 2$
$x = -1,\ m = 1$

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$\boxed{f(x) = d(x)q(x) + r(x)}$$

$f(x) = d(x)\, q(x)$

where $r(x) = 0$ *or* the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, then $d(x)$ *divides evenly* into $f(x)$.

The division algorithm can also be written as

Example 2: Divide $f(x) = (x^4 - 5x^2 + 4)$ by $(x^2 - 1)$.

Example 3: Divide $f(x) = -2 + 3x - 5x^2 + 4x^3 + 2x^4$ by $x^2 + 2x - 3$

$$
\begin{array}{r|l}
 & 2x^2 + 1 \quad \leftarrow \text{quotient} \\
x^2 + 2x - 3 & 2x^4 + 4x^3 - 5x^2 + 3x - 2 \quad \text{dividend} \\
\text{divisor} & 2x^4 + 4x^3 - 6x^2 \\
 & 0 + 0 + x^2 + 3x - 2 \quad \text{new dividend} \\
 & x^2 + 2x - 3 \\
 & 0 + x + 1 \\
 & \text{remainder}
\end{array}
$$

$$\frac{2x^4}{x^2} = 2x^2$$

$$2x^2(x^2 + 2x - 3) = 2x^4 + 4x^3 - 6x^2$$

$$-5x^2 - (-6x^2) = -5x^2 + 6x^2 = x^2$$

Calculus

$f(x) = d(x)\,q(x) + r(x)$

$$\boxed{f(x) = (x^2 + 2x - 3)(2x^2 + 1) + (x + 1)}$$

$$
\begin{array}{r|l}
 & 12 \rightarrow \text{quotient} \\
5 & 64 \rightarrow \text{dividend} \\
\text{divisor} & 5 \\
 & 14 \\
 & 10 \\
 & 4 \text{ remainder}
\end{array}
$$

$64 = 5 \cdot 12 + 4$

Group Practice: Next Written HW

1. Divide the polynomial $f(x) = (x^3 - 9x^2 + 26x - 24)$ by $d(x) = (x - 2)$.

   (a) Does $x - 2$ divide evenly in $f(x)$?

   (b) Calculate the value of $f(2)$.

2. Divide $f(x) = (x^4 - 5x^3 + 2x^2 + 8x - 6)$ by $d(x) = (x - 3)$

   (a) Does $x - 3$ divide evenly in $f(x)$?

   (b) Calculate the value of $f(3)$.

The Remainder and Factor Theorem

Remainder Theorem:
If a polynomial $f(x)$ is divided by $x-a$, then the remainder is equal to $f(a)$.

Factor Theorem:
A polynomial $f(x)$ has a factor $(x-a)$ if and only if $f(a)=0$.

3.4 The Rational Zero Test

What if we are not given any roots or zeros of the polynomial but we are still asked to factor it. For example, in Example 1 we were told to divide $f(x)=x^3-3x^2+4$ by $x-2$ and in doing that we were able to completely factor out $f(x)$. Next, we are interested in how to find potential zeros. This is the result of The Rational zero Test.

The Rational Zero Test

If the polynomial

$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$

has integer coefficients, then every rational zero of f has the form

Rational zero $= \frac{p}{q}$

where p and q have no common factors other than 1, p is a factor of the constant term a_0, and
q is a factor of the leading coefficient a_n.

Possible rational zeros =

Example 4: Find the rational zeros of

1. $f(x) = x^3 - 3x^2 + 4.$

2. $h(x) = 2x^3 + 3x^2 - 8x + 3$

3. $g(x) = x^3 - 2$

2.5 The Fundamental Theorem of Algebra

A nth degree polynomial can have at most n real roots.
In the complex number system , every nth degree polynomial has exactly n zeros.

Types of Zeros of a polynomial function f

- Real

- Complex

Fundamental Theorem of Algebra

Every polynomial function with degree $n \geq 1$ can be factored into n linear factors of the following form:
$f(x) = a(x - r_1) \cdot (x - r_2) \cdot (x - r_3) \cdot ... \cdot (x - r_n)$
where $r_1, r_2, r_3, ..., r_n$ are complex numbers and $a \neq 0$.

Example 5: Completely factor the polynomial over the complex numbers.
$f(x) = (x - 1)(x^2 + 4)(x^2 - 3)$

Example 6: List all possible rational zeros. Find all zeros and completely factor f over the complex numbers.
$f(x) = 2x^4 + x^3 + 17x^2 + 9x - 9$