4. The augmented Matrix of a Linear System

We can write a system of linear equations as a matrix, called the augmented matrix of the system, by writing only the coefficients and constants that appear in the equations.

Linear system	Augmented matrix
$\begin{cases} 3x - 2y + z = 5 \\ x + 3y - z = 0 \\ -x + 4z = 11 \end{cases}$	$\begin{bmatrix} 3 & -2 & 1 & 5 \\ 1 & 3 & -1 & 0 \\ -1 & 0 & 4 & 11 \end{bmatrix}$

Operations we can do on the augmented matrix such that we don't change the system are:

1) swapping two rows
2) multiplying a row a nonzero constant.
3) add/substract rows

5. Solving systems of Linear Equation using Matrices

In general, to solve a system of linear equations using its augmented matrix, we use elementary row operations to arrive at a matrix in a certain form. This form is described in the following box.

ROW-ECHELON FORM AND REDUCED ROW-ECHELON FORM OF A MATRIX

A matrix is in **row-echelon form** if it satisfies the following conditions.

1. The first nonzero number in each row (reading from left to right) is 1. This is called the **leading entry**.
2. The leading entry in each row is to the right of the leading entry in the row immediately above it.
3. All rows consisting entirely of zeros are at the bottom of the matrix.

A matrix is in **reduced row-echelon** form if it is in row-echelon form and also satisfies the following condition.

4. Every number above and below each leading entry is a 0.

Examples:

Row-echelon form

$$\begin{bmatrix} 1 & 3 & -6 & 10 & 0 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$0x+0\cdot y+0\cdot z+1t=\frac{1}{2}$

Leading 1's shift to the right in successive rows

Reduced row-echelon form

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading 1's have 0's above and below them

How to find the row-echelon form of a matrix:
Here is a systematic way to put a matrix in row-echelon form using elementary row operations:

- Start by obtaining 1 in the top left corner. Then obtain zeros below that 1 by adding appropriate multiples of the first row to the rows below it.
- Next, obtain a leading 1 in the next row, and then obtain zeros below that 1.

Find the row-echelon form for the matrix below.

$$\begin{bmatrix} 4 & 8 & -4 & 4 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{bmatrix}$$

calc - choose ref([A])

$$\text{ref}(A) = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$1 \cdot x + 2y + (-1)z) = 1$

$0x + 1 \cdot y + 2z = -3$

$0x + 0y + 1z = -2$

$y + 2(-2) = -3$

$y = 1$

$x + 2(1) + (-1)(-2) = 1$

$x + 4 = 1$

$x = -3$

Back substitution

$(x, y, z) = (-3, 1, -2)$

$$\text{rref}([A]) = \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Gaussian Elimination

ref

row-echelon-form

Find the reduced row-echelon form for the matrix below.

$$\begin{bmatrix} 4 & 8 & -4 & 4 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{bmatrix}$$

Gauss-Jordan Elimination

rref (A)

reduced row-echelon form

Note about inconsistent and consistent systems:

No solution	One solution	Infinitely many solutions
$\begin{bmatrix} 1 & 2 & 5 & 7 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 6 & -1 & 3 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 8 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
Last equation says $0 = 1$	Each variable is a leading variable	z is not a leading variable

$0x+0y+0z=1$

$0=1$

False

No solution

unique solution

infinitely many solutions

z is free

z can be anything

Practice: Solve the system of equations.

$$\begin{cases} 4x - 3y + z = -31 \\ -2x + y - 3x = 9 \\ x - y + 2z = -8 \end{cases}$$

$$A = \begin{bmatrix} 4 & -3 & 1 & -31 \\ -5 & 1 & 0 & 9 \\ 1 & -1 & 2 & -8 \end{bmatrix}$$

$$rref(A) = \begin{bmatrix} 1 & 0 & 0 & .5 \\ 0 & 1 & 0 & 11.5 \\ 0 & 0 & 1 & 1.5 \end{bmatrix}$$

Determine whether the system is inconsistent or dependent. If it is dependent, find the complete solution. Express your answer in terms of t, where $x = x(t)$, $y = y(t)$, and $z = t$.

$$\begin{cases} 2x - 3y - 9z = -5 \\ x + 3z = 2 \\ -3x + y - 4z = -3 \end{cases}$$

$$A = \begin{bmatrix} 2 & -3 & -9 & -5 \\ 1 & 0 & 3 & 2 \\ -3 & 1 & -4 & -3 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\to 1 \cdot x + 0y + 3z = 2$

$\to 0x + 1 \cdot y + 5z = 3$

infinitely many solution

z is anything

$z = t$

$y = 3 - 5z$

$\boxed{y = 3 - 5t}$

$\boxed{z = t}$

$\boxed{x = 2 - 3t}$

$x = 2 - 3z$

$x = 2 - 3t$

$\boxed{(2 - 3t,\ 3 - 5t,\ t)}$

infinitely many sol.

$t = 0$

$(2, 3, 0)$

$t = 1$

$(-1, -2, 1)$