Day 4: MTH 144 : Precalculus

Combining functions and Inverses

1. Adding, subtracting, multiplying and dividing functions

Two functions f and g can be combined to form new functions $f+g$, $f-g$, fg, and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers.

ALGEBRA OF FUNCTIONS

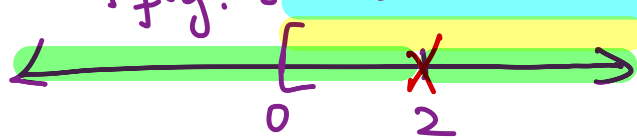
Let f and g be functions with domains A and B. Then the functions $f+g$, $f-g$, fg, and f/g are defined as follows.

$(f+g)(x) = f(x) + g(x)$	Domain $A \cap B$
$(f-g)(x) = f(x) - g(x)$	Domain $A \cap B$
$(fg)(x) = f(x)g(x)$	Domain $A \cap B$
$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$	Domain $\{x \in A \cap B \mid g(x) \neq 0\}$

Practice: Perform the following function operations for f and g and find the domains of the resulting functions.

$f(x) = \dfrac{1}{x-2}$ and $g(x) = \sqrt{x}$.

Domain $\begin{cases} f: (-\infty, 2) \cup (2, \infty) \\ g: [0, \infty) \\ f+g: [0,2) \cup (2, \infty) \end{cases}$



(a) $(f+g)(x) = \dfrac{1}{x-2} + \sqrt{x}$

(b) $(f-g)(x) = \dfrac{1}{x-2} - \sqrt{x}$

$f-g$ has domain : $[0,2) \cup (2,\infty)$

(c) $(fg)(x) = f(x)g(x) = \dfrac{1}{x-2} \cdot \sqrt{x} = \dfrac{\sqrt{x}}{x-2}$ domain fg: $[0,2) \cup (2,\infty)$

(d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{\frac{1}{x-2}}{\sqrt{x}} = \dfrac{1}{\sqrt{x}(x-2)}$ domain: $(0,2) \cup (2,\infty)$

2. Composition of Functions

Another way to combine two functions f and g is to compose them. Composing the function $f(x)$ with the function $g(x)$, denoted $(f \circ g)(x)$, means that the output of g becomes the input of f. In other words $(f \circ g)(x) = f(g(x))$ or the function f is evaluated at $g(x)$.

Example: Let $f(x) = \dfrac{1}{x+2}$ and $g(x) = \dfrac{4}{x-1}$. Find the following:

(a) $f \circ g$ and its domain

$$(f \circ g)(x) = f(g(x)) = \frac{1}{g(x)+2} = \frac{1}{\frac{4}{x-1} + \frac{2}{1}\cdot\frac{(x-1)}{(x-1)}} = \frac{1}{\frac{4+2x-2}{x-1}}$$

$$= \frac{1}{\frac{2+2x}{x-1}} = \boxed{1 \cdot \frac{x-1}{2x+2}}$$

Domain: $x \neq -1$, $x \neq 1$

-1 $\quad$ 1

$(-\infty,-1) \cup (-1,1) \cup (1,\infty)$

(b) $g \circ f$ and its domain

$$(g \circ f)(x) = g(f(x)) = \frac{4}{f(x)-1} = \frac{4}{\frac{1}{x+2} - \frac{1}{1}\cdot\frac{x+2}{(x+2)}} = \frac{4}{\frac{1-x-2}{x+2}}$$

$$= \frac{4}{\frac{-x-1}{x+2}} = \boxed{4 \cdot \frac{x+2}{-x-1}}$$

Domain: $x \neq -1$, $x \neq -2$

-2 $\quad$ -1

$(-\infty,-2) \cup (-2,-1) \cup (-1,\infty)$

!

NOTE The domain of $f \circ g$ is the intersection of the domain of inner function g and the resulting function $f \circ g$.

Practice: Exercise 1

Let $f(x) = \dfrac{x}{x+3}$ and $g(x) = 8x - 3$. Find the following

1. $(f \circ g)(x)$ and its domain

$$(f \circ g)(x) = f(g(x)) = \frac{g(x)}{g(x)+3} = \frac{8x-3}{8x-3+3} = \frac{8x-3}{8x}$$

Domain: $x \neq 0$ so: $(-\infty, 0) \cup (0, \infty)$

2. $(g \circ f)(x)$ and its domain

$$(g \circ f)(x) = g(f(x)) = 8f(x) - 3 = 8\left(\frac{x}{x+3}\right) - 3 = \frac{8x}{x+3} - 3\,\frac{x+3}{x+3}$$

$$= \frac{8x - 3(x+3)}{x+3} = \frac{5x-9}{x+3};$$ Domain: $x \neq -3$ $(-\infty, -3) \cup (-3, \infty)$

Practice: Exercise 2

Let $f(x) = x^3 + 8$ and $g(x) = x - 5$ and $h(x) = \sqrt{x}$. Find $f \circ g \circ h$.

$$(f \circ g \circ h)(x) = f(\underbrace{g(h(x))}) = f(\,h(x) - 5) =$$

$$= f(\sqrt{x} - 5) = (\sqrt{x} - 5)^3 + 8$$

Practice: Exercise 3

A spherical balloon is being inflated. The radius of the balloon is increasing at a rate of 3cm/s.

1. Find a function $\underline{f}$ that models the radius as a function of time t, in seconds.

$$f(t) = 3t$$

2. Find a function g that models the volume as a function of the radius r, in cm.

$$g(r) = \frac{4\pi r^3}{3}$$ (formula of volume of a sphere)

3. Find and interpret $g \circ f$. $(g \circ f)(t) = g(f(t)) = \dfrac{4\pi(3t)^3}{3} = 36\pi t^3$

finds the volume in term of time.

6. One-to-one Functions

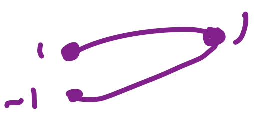
A function f is **one-to-one** if there is a one-to-one correspondence between the inputs and the outputs. In other words, f never takes on the same values twice.

Examples: Is f one-to-one?

$f(x) = x^2$

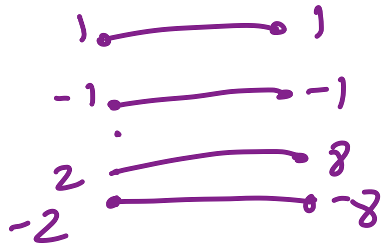
$f(1) = 1$

$f(-1) = 1$



NOT

$f(x) = x^3$



yes

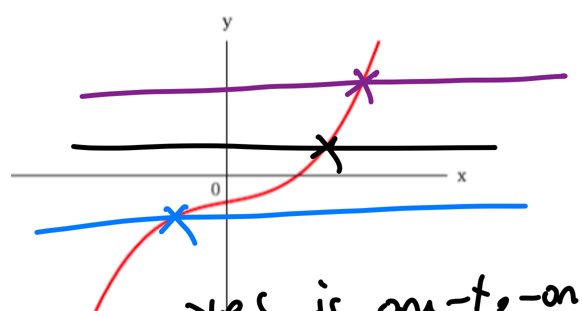
$f(1) = 1$

$f(-1) = -1$

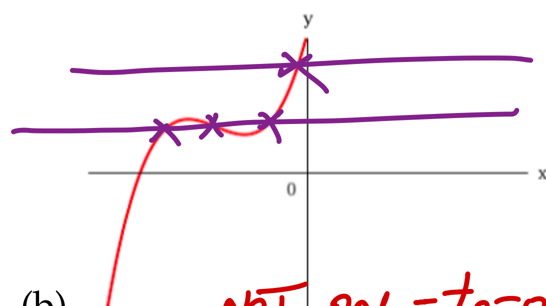
$f(-2) = -8$

$f(2) = 8$

How to determine if a function f is one-to-one - Graphically
Determine if the functions f and g are one-to-one. Horizontal line test



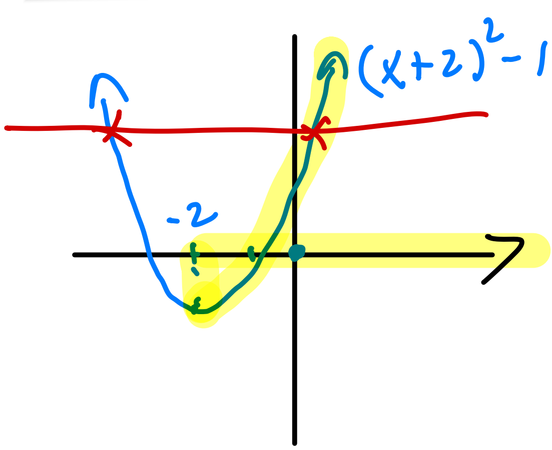
(a) yes is one-to-one



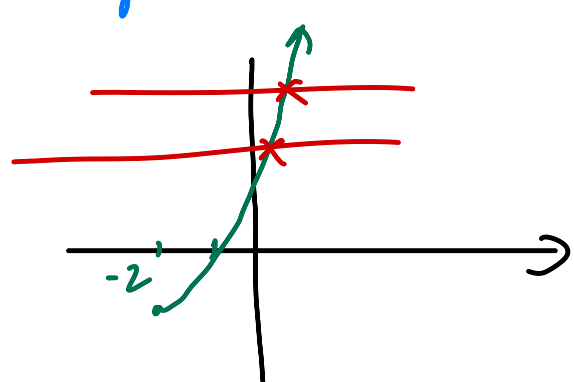
(b) NOT one-to-one

Practice - Restricted Domain
Graph the function $f(x) = (x + 2)^2 - 1$ and decide if it is one-to-one on its entire domain.

x^2 shifted left 2 units and down 1 unit



Domain $(-\infty, \infty)$



Restricted domain: $[-2, \infty)$

$x \xrightarrow{f} f(x)=y$, $\quad y \xrightarrow{f^{-1}} x$

f^{-1} means the inverse of f

7. Inverse Functions

A one-to-one function with domain A and range B has an **inverse function** which takes on as domain the range of f and gives as output the domain of f. In other words the inverse of a function f undoes what f does. The inverse, denoted as f^{-1} takes as input the output of f and gives as output the input of f, i.e. if $f(x) = y$ then $f^{-1}(y) = x$.

For example if f is the function that takes the input x multiplies it by 5, adds 2 then takes the 5th power of the result, what would the inverse f^{-1} have to be to undo what f did?

$$f(x) = (5x+2)^5$$

$$f^{-1}(x) = \frac{\sqrt[5]{x} - 2}{5} \quad !$$

the opposite in reverse order

The inverse function f^{-1} reverses the effect of f!

INVERSE FUNCTION PROPERTY

Let f be a one-to-one function with domain A and range B. The inverse function f^{-1} satisfies the following cancellation properties:

$(f^{-1} \circ f)(x)$ $\qquad f^{-1}(f(x)) = x \quad$ for every x in A

$(f \circ f^{-1})(x)$ $\qquad f(f^{-1}(x)) = x \quad$ for every x in B

Conversely, any function f^{-1} satisfying these equations is the inverse of f.

Practice: Are f and g inverses of each other?

$$f(x) = 8 - 4x, \quad g(x) = \frac{8-x}{4}$$

Check: $(f \circ g)(x) \stackrel{?}{=} x$

$(g \circ f)(x) \stackrel{?}{=} x$

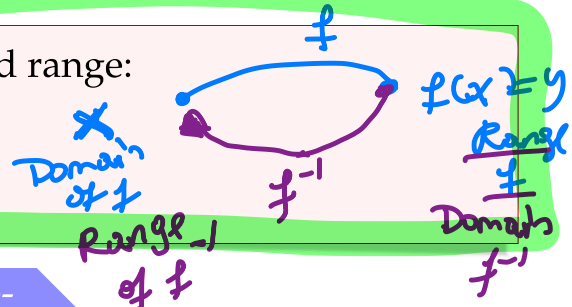
$$(f \circ g)(x) = f(g(x)) = 8 - 4\,g(x) = 8 - \cancel{4}\left(\frac{8-x}{\cancel{4}}\right) = \cancel{8} - \cancel{8} + x = x$$

$$(g \circ f)(x) = g(f(x)) = \frac{8 - f(x)}{4} = \frac{8 - (8-4x)}{4} =$$

$$= \frac{\cancel{8} - \cancel{8} + 4x}{4} = \frac{4x}{4} = x$$

yes!

NOTE: The function and its inverse switch domain and range:
Domain of f = Range of f^{-1}
Range of f = Domain of f^{-1}



8. How to find the inverse of a one-to-one function - Algebraically

Let f be a one-to one function. Then $f(x) = y$ for any x in the domain of f and f has an inverse denoted f^{-1}.

Example: Let $f(x) = \dfrac{x-9}{x+9}$.

(a) Is f one-to-one?

yes (posse HTL)

(b) Find the domain of f.

$x \neq -9 \quad (-\infty, -9) \cup (-9, \infty)$

(c) Can you find the range of f without graphing it?

Not really

(d) Find the inverse of f.

$$f^{-1}(x) = \frac{-9-9x}{x-1}$$

Domain f^{-1} : $x \neq 1$

$(-\infty, 1) \cup (1, \infty)$

Steps to find f^{-1}

1) write $f(x) = \dfrac{x-9}{x+9}$

or $y = \dfrac{x-9}{x+9}$

2) Switch x and y

$$\boxed{x = \frac{y-9}{y+9}}$$

3) Solve for y

$$x(y+9) = y-9$$
$$xy + 9x = y-9$$
$$xy - y = -9-9x$$
$$y(x-1) = -9-9x$$
$$y = \frac{-9-9x}{x-1} = f^{-1}(x)$$

(e) Can you find the range of f now using the information you have about f^{-1}?

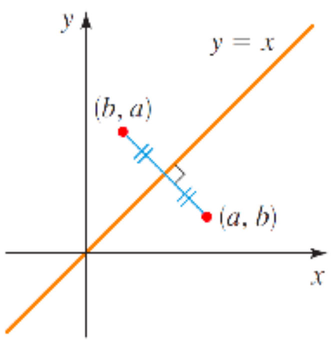
Range of f = domain of f^{-1}

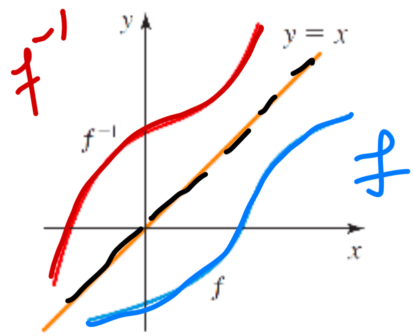
$(-\infty, 1) \cup (1, \infty)$

9. How to find the inverse of a one-to-one function - Graphically

Let f be a one-to one function. Then $f(x) = y$ for any x in the domain of f ~~and f~~ has an inverse f^{-1}. A point (a, b) on the graph of f is going to become (b, a) on the graph of f^{-1} since the input is switched with the output.

The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.





Example: Given the graph of f below find the graph of f^{-1}.