Mth 146: Day 10
Sequences and Series Review

1 Why sequences and series?

Much of Calculus II is about one idea: *adding up infinitely many numbers* and deciding whether the total makes sense. This is how calculators compute e^x, $\sin x$, and $\ln x$ (Taylor series), and how we approximate functions with polynomials. Today we review the precalculus language of sequences and series and then add the calculus tool that makes everything work: the **limit**.

2 Sequences and notation

Roughly speaking, a sequence is an infinite list of numbers written in a definite order.

Definition. A **sequence** is a

____

(sometimes $\{0, 1, 2, \dots\}$). The numbers in the list are called the ____________ of the sequence.

Notation:

$$a_1,\ a_2,\ a_3,\ \dots,\ a_n,\ \dots \qquad\qquad \{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}$$

a_n is called the ________________________. Since $a_n = f(n)$ for a function f, we can *graph* a sequence as isolated dots (n, a_n).

Example 1. Find the first three terms and the 100th term. Then write the sequence using { } notation.

(a) $a_n = 2n - 1$

(c) $c_n = \dfrac{n}{n+1}$

(b) $b_n = n^2 - 1$

(d) $d_n = \dfrac{(-1)^{n+1}}{n}$

Example 2. Find a formula for the general term a_n, assuming the pattern continues. (Start at $n = 1$.)

(a) $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \cdots$

(c) $1,\ \frac{1}{2},\ \frac{1}{6},\ \frac{1}{24},\ \frac{1}{120},\ \cdots$

(b) $-2,\ 4,\ -8,\ 16,\ -32,\ \cdots$

(d) $\frac{1}{2},\ -\frac{1}{4},\ \frac{1}{8},\ -\frac{1}{16},\ \cdots$

Definition. An **alternating sequence** is a sequence whose ________________.

Tip: $(-1)^n$ makes the ______-numbered terms negative;
$(-1)^{n+1}$ makes the ______-numbered terms negative.

Recursively defined sequences. When the nth term depends on some or all of the terms before it, the sequence is ________________.

Example 3. Find the first five terms: $a_1 = 4$ and $a_n = 2a_{n-1} + 1$ for $n > 1$.

3 Factorials

Factorials appear in Taylor series and are the key to the Ratio Test. The skill you need in Calc II is *simplifying* ratios of factorials.

Definition. For a positive integer n,
$n! =$ ____________________________,
and by definition $0! =$ _____.
$5! =$ ____________________

Key identities: $(n+1)! =$ __________
$(2n+2)! =$ __________________

Example 4. Simplify.

(a) $\dfrac{8!}{3! \cdot 5!}$

(b) $\dfrac{n!}{(n+2)!}$

(c) $\dfrac{(2n+2)!}{(2n)!}$

(d) $\dfrac{(n+1)!}{(n+1)^{n+1}} \cdot \dfrac{n^n}{n!}$ (a Ratio Test preview)

This is the new, Calc II part. What happens to a_n as $n \to \infty$?

Definition. We write $\lim_{n\to\infty} a_n = L$ if we can make the terms a_n as close to L as we like by ________________________. If the limit exists (and is a finite number) the sequence __________; otherwise it __________.

Theorem (sequences from functions). If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ for every integer n, then $\lim_{n\to\infty} a_n = L$.

Why it matters: we can use all of our Calc I tools on sequences, including ____________.

Warning: the converse is false. $a_n = \sin(\pi n) = 0$ for all n, so $a_n \to 0$, but $\lim_{x\to\infty} \sin(\pi x)$ _______.

A useful fact: $\displaystyle\lim_{n\to\infty} \frac{1}{n^r} =$ ______ if $r > 0$.

Limit Laws for Sequences. Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant. Then

1. **Sum Law**
   $$\lim_{n\to\infty} (a_n + b_n) = \underline{\hspace{6cm}}$$
2. **Difference Law**
   $$\lim_{n\to\infty} (a_n - b_n) = \underline{\hspace{6cm}}$$
3. **Constant Multiple Law**
   $$\lim_{n\to\infty} c\, a_n = \underline{\hspace{6cm}}$$
4. **Product Law**
   $$\lim_{n\to\infty} (a_n b_n) = \underline{\hspace{6cm}}$$
5. **Quotient Law**
   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \underline{\hspace{6cm}} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$
6. **Power Law**
   $$\lim_{n\to\infty} a_n^p = \underline{\hspace{6cm}} \quad \text{if } p > 0 \text{ and } a_n > 0$$

Squeeze Theorem. If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim a_n = \lim c_n = L$, then $\lim b_n = L$.

Corollary. If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n =$ _____.

Continuity. If $\lim a_n = L$ and f is continuous at L, then $\lim\limits_{n\to\infty} f(a_n) =$ ______.

A limit you will use constantly:

$$\lim_{n\to\infty} r^n = \begin{cases} \underline{\hspace{2cm}} & \text{if } |r| < 1 \\ \underline{\hspace{2cm}} & \text{if } r = 1 \\ \underline{\hspace{4cm}} & \text{if } r \le -1 \text{ or } r > 1 \end{cases}$$

So $\{r^n\}$ converges exactly when __________.

Example 5. Determine whether each sequence converges or diverges. If it converges, find the limit.

(a) $a_n = \dfrac{3n^2 - n}{5n^2 + 4}$

(b) $a_n = \dfrac{\ln n}{n}$

(c) $a_n = \dfrac{(-1)^n}{n}$

(d) $a_n = (-1)^n$

(e) $a_n = \sin\left(\frac{\pi}{n}\right)$

(f) $a_n = \sqrt{n+1} - \sqrt{n}$

(g) $a_n = \frac{n!}{n^n}$

Sometimes we can prove a sequence converges *without* knowing its limit.

Definition. $\{a_n\}$ is **increasing** if __________ for all n.

Definition. $\{a_n\}$ is **decreasing** if __________ for all n.

Definition. $\{a_n\}$ is **monotonic** if it is ______________________.

Definition. $\{a_n\}$ is **bounded above** if there is a number M with _______ for all n.

Definition. $\{a_n\}$ is **bounded below** if there is a number m with _______ for all n.

Definition. $\{a_n\}$ is **bounded** if it is ______________________.

Three ways to check monotonic:

(1) sign of $a_{n+1} - a_n$

(2) compare $\dfrac{a_{n+1}}{a_n}$ to 1 (positive terms)

(3) sign of $f'(x)$ where $f(n) = a_n$

Monotonic Sequence Theorem. Every ___________, ___________ sequence is convergent.

Example 6. Show that $a_n = \dfrac{n}{n^2+1}$ is decreasing and bounded. Does it converge?

6 Reflection

For each item below, rate your comfort level and jot a quick note: What clicked? What's still fuzzy? What would help?

1. **Find terms, general-term formulas, and limits of sequences (including recursively defined ones).** 1 2 3 4 5

2. **Use the Monotonic Sequence Theorem to show a sequence converges, even without knowing its limit.** 1 2 3 4 5