Mth 146: Day 6
7.4 Integration of Rational Functions by Partial Fractions

1 Part 1: New Concepts & Worked Examples

The method of Partial Fractions allows us to integrate a rational function $\frac{P(x)}{Q(x)}$, by first expressing it as a sum of simpler fractions called **Partial Fractions** that can be integrated more easily.

For example:

$$\int \frac{1}{x^2+3x-4} = \int \frac{1}{5(x-1)} - \int \frac{1}{5(x+4)}$$

$$\int \frac{1}{5(x-1)}\,dx$$

The process of breaking down a **proper** rational function into more fractions is called **Partial Fraction Decomposition**.

Recall: A proper rational function is a function where the degree of the numerator is strictly less than the degree of denominator.

Ex ① $\frac{x^2+4}{2x^2+x+1}$ not proper

② $\frac{x^3+2x}{x^2+1}$ not proper

③ $\frac{x+1}{2x^2+3}$ yes

Recal Trig sub

$\sqrt{a^2-x^2} \longrightarrow x = \sin\theta$

$\sqrt{a^2+x^2} \longrightarrow x = \tan\theta$

$\sqrt{x^2-a^2} \longrightarrow x = \sec\theta$

Ex $\displaystyle\int \frac{x}{\sqrt{9-4x^2}}\,dx$

$\sqrt{3^2-(2x)^2} \longrightarrow$ sub $2x = 3\sin\theta$

$2\,dx = 3\cos\theta\,d\theta$

$dx = \frac{3}{2}\cos\theta\,d\theta$

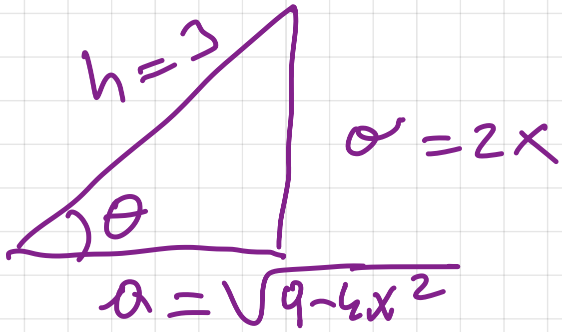
$$\int \frac{\frac{3}{2}\sin\theta\ \frac{3}{2}\cos\theta\,d\theta}{\sqrt{9-(3\sin\theta)^2}} =$$

$$= \frac{9}{4}\int \frac{\sin\theta\cos\theta\,d\theta}{\sqrt{9-9\sin^2\theta}} = \frac{9}{4}\int \frac{\sin\theta\cos\theta\,d\theta}{\sqrt{9(1-\sin^2\theta)}}$$

$$= \frac{9}{4}\int \frac{\sin\theta\cos\theta\,d\theta}{3\sqrt{\cos^2\theta}} = \frac{9}{4}\cdot\frac{1}{3}\int \frac{\sin\theta\cancel{\cos\theta}\,d\theta}{\cancel{\cos\theta}}$$

$$= \frac{3}{4}\int \sin\theta\,d\theta = \frac{3}{4}(-\cos\theta) = -\frac{3}{4}\cos\theta$$

$$= -\frac{\cancel{3}}{4}\,\frac{\sqrt{9-4x^2}}{\cancel{3}} + C = \boxed{-\frac{1}{4}\sqrt{9-4x^2}+C}$$



$$2x = 3\sin\theta$$

$$\frac{o}{h} = \sin\theta = \frac{2x}{3}$$

$$a^2 + o^2 = h^2$$

$$a^2 + 4x^2 = 9$$

$$a = \sqrt{9-4x^2}$$

$$\cos\theta = \frac{a}{h} = \frac{\sqrt{9-4x^2}}{3}$$

Case 1: The denominator is a product of distinct linear factors

Example 1. $\int \frac{1}{x^2+3x-4}\,dx$

proper so we do PFD

$$\frac{1}{x^2+3x-4} = \frac{1}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1} \qquad (x+4)(x-1)$$

clear denominator

$$1 = \frac{A}{\cancel{x+4}}\cancel{(x+4)}(x-1) + \frac{B}{\cancel{x-1}}(x+4)\cancel{(x-1)}$$

$$1 = A(x-1) + B(x+4)$$

$$0x + 1 = Ax - A + Bx + 4B$$

$$0x + 1 = x(A+B) - A + 4B$$

$A+B=0 \Rightarrow A=-B \Rightarrow A = -1/5$

$-A+4B=1 \Rightarrow -(-B)+4B=1 \Rightarrow 5B=1 \Rightarrow B=1/5$

$$\int \frac{1}{x^2+3x-4}\,dx = \int \frac{-1/5}{x+4} + \frac{1/5}{x-1}\,dx$$

$$= -\frac{1}{5}\int \frac{1}{x+4}\,dx + \frac{1}{5}\int \frac{1}{x-1}\,dx$$

$$= -\frac{1}{5}\ln|x+4| + \frac{1}{5}\ln|x-1| + C$$

Case 2: The denominator is a product of linear factors some of which are repeated

Example 2. $\int_{-1}^{0} \left(\frac{x^3}{x^2-2x+1}\right) dx.$

$\frac{x^3}{x^2-2x+1}$ not proper. Do long division first.

$$\begin{array}{r|l} & x+2 \\ \hline x^2-2x+1 & x^3+0x^2+0x \\ & x^3-2x^2+x \\ \hline & 0 \quad 2x^2-x+0 \\ & \quad 2x^2-4x+2 \\ \hline & \quad 0 \quad 3x-2 \end{array}$$

$$\frac{x^3}{x^2-2x+1} = x+2 + \frac{3x-2}{x^2-2x+1}$$

$$\frac{x^3}{x^2-2x+1} = x+2 + \frac{3x-2}{(x-1)^2}$$

PFD for this one

$$\cancel{(x-1)^2}\,\frac{3x-2}{\cancel{(x-1)^2}} = \left(\frac{A}{x-1} + \frac{B}{(x-1)^2}\right)(x-1)^2 \quad \text{clear denominators}$$

$$3x-2 = A(x-1) + B$$

$\Rightarrow 3x - 2 = Ax - A + B$

$A = 3$

$-2 = -A + B$

$-2 = -3 + B \Rightarrow B = 1$

$$\frac{3x-2}{(x-1)^2} = \frac{3}{x-1} + \frac{1}{(x-1)^2}$$

$$\boxed{\int_{-1}^{0} \frac{x^3}{x^2-2x+1}\,dx} = \int_{-1}^{0} \left(x+2+\frac{3}{x-1}+\frac{1}{(x-1)^2} \right) dx$$

$$= \left(\quad \Big| \quad \right) - \left(\frac{1}{2} - 2 + 3\ln 2 + \frac{1}{2} \right) = \boxed{2 - 3\ln 2}$$

$$\int x\,dx + 2\int dx + 3\int \frac{1}{x-1}\,dx + \underbrace{\int \frac{1}{(x-1)^2}\,dx}_{*}$$

$$= \frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1}$$

$*$ $u = x-1$

$du = dx$

$$\int \frac{1}{(x-1)^2}\,dx = \int \frac{1}{u^2}\,du = \int u^{-2}\,du = \frac{u^{-1}}{-1} = -\frac{1}{u} = -\frac{1}{x-1}$$

Case 3: Denominator contains non-repeating irreducible quadratic factors $ax^2 + bx + c$ where $b^2 - 4ac < 0$

$$\require{enclose}x^2+4 \enclose{longdiv}{x^3+4}\overset{x}{}$$
$$\underline{x^3+4x}$$
$$0 \quad -4x+4$$

Example 3. $\int \frac{x^3 + 4}{x^2 + 4} dx.$

$$= \int x + \frac{-4x+4}{x^2+4} dx = \int x\,dx - 4\int \frac{x}{x^2+4} dx + 4\int \frac{1}{x^2+4} dx$$

$$= \frac{x^2}{2} - 4 \cdot \frac{1}{2} \ln|x^2+4| + 4\frac{1}{2} \arctan\frac{x}{2} + C$$

$$\frac{x^3+4}{x^2+4} = x + \frac{-4x+4}{x^2+4}$$

$$\int \frac{x}{x^2+4} dx = \frac{1}{2}\int \frac{1}{u} du = \frac{1}{2}\ln|x^2+4|$$

$u = x^2+4$

$du = 2x\,dx$ $\quad x\,dx = \frac{1}{2}du$

Example 4. $\int \frac{10}{(x - 1)(x^2 + 9)} dx.$

$$\frac{10\,(x-1)(x^2+9)}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9} \Big/ (x-1)(x^2+9)$$

clear denomiator

$$10 = A(x^2+9) + (Bx+C)(x-1)$$

$$10 = Ax^2 + 9A + Bx^2 - Bx + Cx - C$$

$$0x^2 + 0x + 10 = (A+B)x^2 + x(C-B) + 9A - C$$

$0 = A + B \Rightarrow A = -B$

$0 = C - B \Rightarrow C = B$

$10 = 9A - C \Rightarrow 10 = 9(-B) - B$ $\quad -9B - B$

$10 = -10B \Rightarrow B = -1$

$C = -1$

$A = 1$

$$\int \frac{10}{(x-1)(x^2+9)}\,dx = \int \frac{1}{x-1}\,dx + \int \frac{-x-1}{x^2+9}\,dx$$

$$= \ln|x-1| - \underbrace{\int \frac{x}{x^2+9}\,dx} - \underbrace{\int \frac{1}{x^2+9}\,dx}$$

$$= \quad . \quad . \quad .$$

Case 4: Denominator contains repeating irreducible quadratic factors

Example 5. $\displaystyle\int \frac{dx}{x(x^2+4)^2}$

$$\frac{1}{x(x^2+4)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

$$1 = A(x^2+4)^2 + (Bx+C)x(x^2+4) + (Dx+E)x$$

$$1 = A(x^4+8x^2+16) + (Bx^2+Cx)(x^2+4) + Dx^2+Ex$$

$$1 = Ax^4 + 8Ax^2 + 16A + Bx^4 + 4Bx^2 + Cx^3 + 4Cx + Dx^2 + Ex$$

$$1 = x^4(A+B) + Cx^3 + x^2(8A+4B+D) + x(4C+E) + 16A$$

$16A = 1 \Rightarrow A = 1/16$

$A+B = 0 \rightarrow A = -B \Rightarrow B = -1/16$

$C = 0$

$8A + 4B + D = 0 \rightarrow 8\left(\frac{1}{16}\right) + 4\left(\frac{-1}{16}\right) + D = 0$

$\frac{1}{4} + D = 0$

$D = -1/4$

$4C + E = 0 \Rightarrow E = 0$

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$A = 1/6$, $B = -1/6$, $C = E = 0$, $D = -1/4$

$$\frac{1}{x(x^2+4)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

$$\int \frac{1}{x(x^2+4)^2}\,dx = \frac{1}{16}\int \frac{1}{x}\,dx - \frac{1}{16}\int \frac{x}{x^2+4}\,dx \cancel{+} - \frac{1}{4}\int \frac{x}{(x^2+4)^2}\,dx$$

$$= \frac{1}{16}\ln|x| - \frac{1}{16}\cdot\frac{1}{2}\ln(x^2+4) - \frac{1}{4}\left(-\frac{1}{2(x^2+4)}\right) + c$$

$$= \frac{1}{16}\ln|x| - \frac{1}{32}\ln(x^2+4) + \frac{1}{2}\,\frac{1}{x^2+4} + C$$

$$\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\int \frac{1}{u}\,du$$

u-sub

$u = x^2+4$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \frac{1}{2}\ln|x^2+4|$$

$$\int \frac{x}{(x^2+4)^2}\,dx = \frac{1}{2}\int \frac{1}{u^2}\,du$$

$u = x^2+4$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \frac{1}{2}\,\frac{u^{-2+1}}{-1}$$

$$= -\frac{1}{2u}$$

Rationalizing Substitutions

Example 6. $\displaystyle\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx.$

next class

Example 7. $\displaystyle\int \frac{dx}{x\sqrt{x+1}}.$