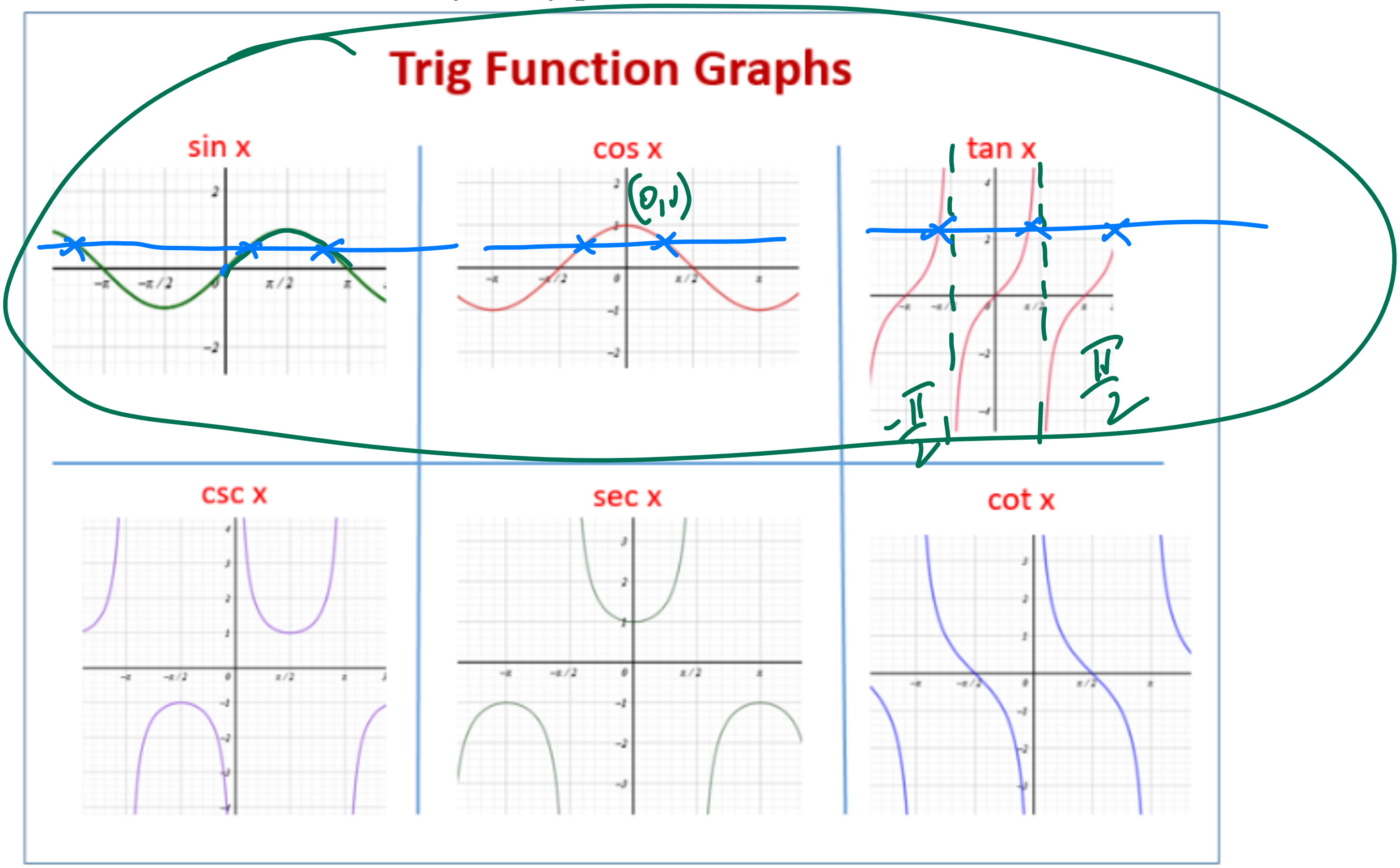
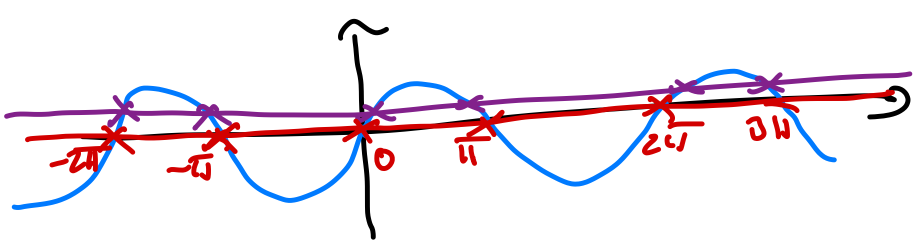
Recall Graphs of Sine, Cosine and Tangent Functions

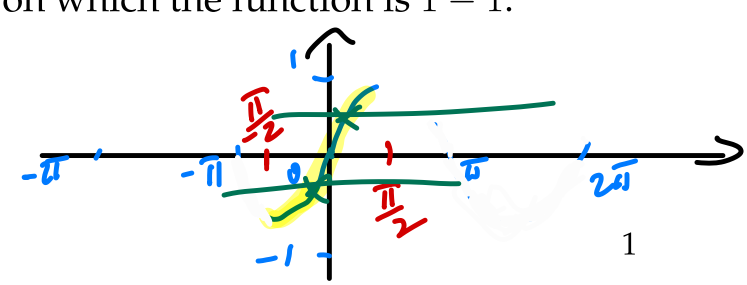
The graphs of the basic trigonometric functions $\sin(x), \cos(x)$, and $\tan(x)$ all fail the horizontal line test at infinitely many points.



For example, $\sin x = 0$ for $x = 2n\pi$ for any integer n.



However, we can deal with this problem by **restricting the domain** of each function to an interval on which the function is $1-1$.



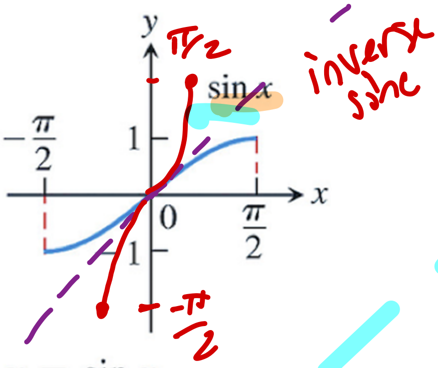
Domain : $(-\infty, \infty)$

Range: $[-1, 1]$

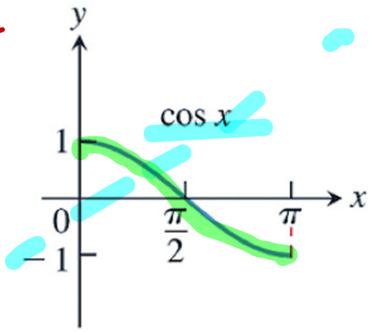
Restricted domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Range $[-1, 1]$

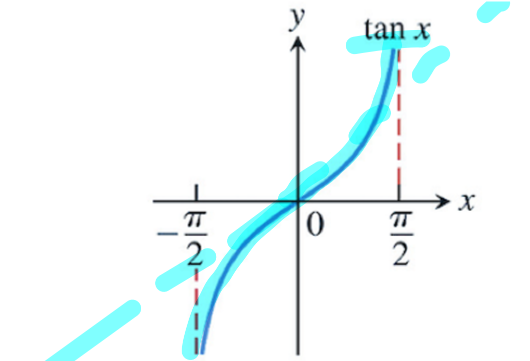
Domain restrictions that make the trigonometric functions one-to-one



$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



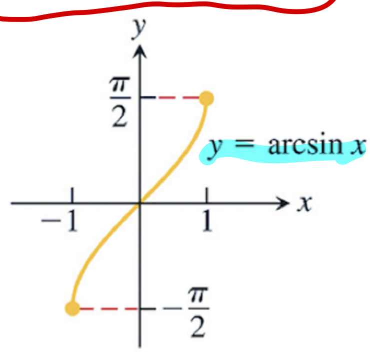
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$

Graphs of inverse trigonometric functions

f with its inverse f^{-1}

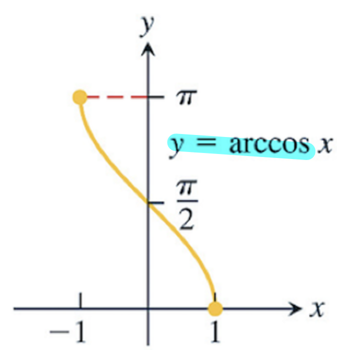
Now that we are dealing with invertible functions, we can define inverses for each of these trig functions. We can denote the inverse of $\sin x$ on its restricted domain by $\sin^{-1}(x)$ or by $\arcsin(x)$.

Domain: $-1 \le x \le 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



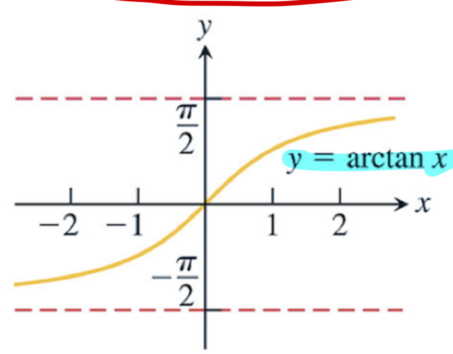
(a)

Domain: $-1 \le x \le 1$
Range: $0 \le y \le \pi$



(b)

Domain: $-\infty < x < \infty$
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



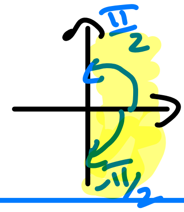
(c)

Example 1. *Calculate the following, paying close attention to the domain and range of our inverse trigonometric functions:*

1. $\arcsin\left(-\frac{1}{2}\right) = ? = -\frac{\pi}{6}$

$\sin ? = -1/2$

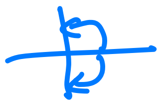
$D: [-1,1]$, $R: [-\frac{\pi}{2}, \frac{\pi}{2}]$



2. $\arcsin(\sqrt{3}/2) = ? = \frac{\pi}{3}$

$D: [-1,1]$, $R\ [-\pi/2, \frac{\pi}{2}]$

$\sin ? = \frac{\sqrt{3}}{2}$



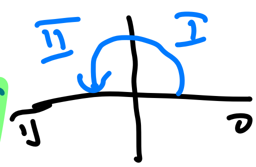
3. $\sin^{-1}(2) = ?$

$D: [-1,1]$, $R\ [-\pi/2, \pi/2]$

not defined

4. $\arccos\left(\frac{\sqrt{2}}{2}\right) = ? = \frac{\pi}{4}$

$D: [-1,1]$, $R: [0, \pi]$

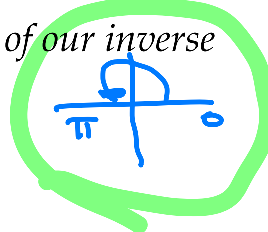


$\cos ? = \frac{\sqrt{2}}{2}$

$\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$

5. $\cos^{-1}(-1) = ? = \pi$

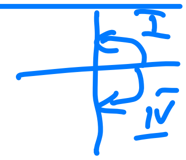
$D: [-1,1]$ $R: [0, \pi]$



$\cos ? = -1$

6. $\arctan 0 = ? = 0$

$D: (-\infty, \infty)$ $R: (-\pi/2, \pi/2)$

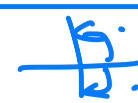


$\tan ? = 0$

$\tan 0 = 0$

7. $\arctan\left(\frac{\sqrt{3}}{3}\right) = ? = \frac{\pi}{6}$

$D: (-\infty, \infty)$ $R: (-\pi/2, \pi/2)$



$\tan ? = \frac{\sqrt{3}}{3}$

Compositions of Functions

$(f \circ g)(x) = f(g(x))$

When we compose trig functions and their inverses (i.e. sine composed with arcsine) we can make use of the inverse properties.

$f(f^{-1}(x)) = x$

$f^{-1}(f(x)) = x$

Inverse Properties

$\sin(\arcsin x) = x$

$\arcsin(\sin x) = x$

If $-1 \leq x \leq 1$ and $-\pi/2 \leq y \leq \pi/2$, then

$$\sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y.$$

If $-1 \leq x \leq 1$ and $0 \leq y \leq \pi$, then

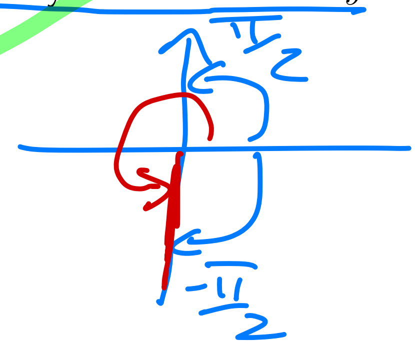
$$\cos(\arccos x) = x \quad \text{and} \quad \arccos(\cos y) = y.$$

If x is a real number and $-\pi/2 < y < \pi/2$, then

$$\boxed{\tan(\arctan x) = x} \quad \text{and} \quad \arctan(\tan y) = y.$$

Caution!!! Keep in mind that these properties do not apply for arbitrary values of x and y. For instance,

$$\arcsin\left(\sin \frac{3\pi}{2}\right) \neq \frac{3\pi}{2}.$$



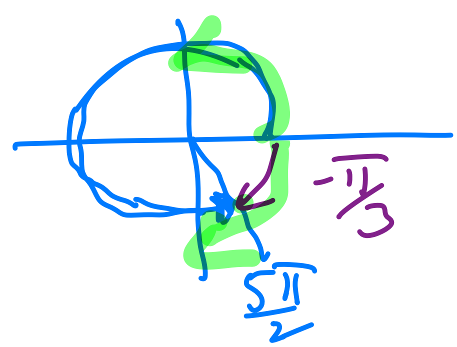
$\arcsin\left(\sin\left(-\frac{\pi}{2}\right)\right) = -\pi/2$

Example 2. *If possible, find the exact value.*

a. $\tan[\arctan(-5)] = -5$

b. $\arcsin\left(\sin \frac{5\pi}{3}\right) = \arcsin\left(\sin\left(-\frac{\pi}{3}\right)\right)$

$= -\pi/3$



c. $\cos(\cos^{-1}\pi)$ $= \cos(\underbrace{\arccos \pi}_{\text{cannot be evaluated}}) = \text{undefined}$

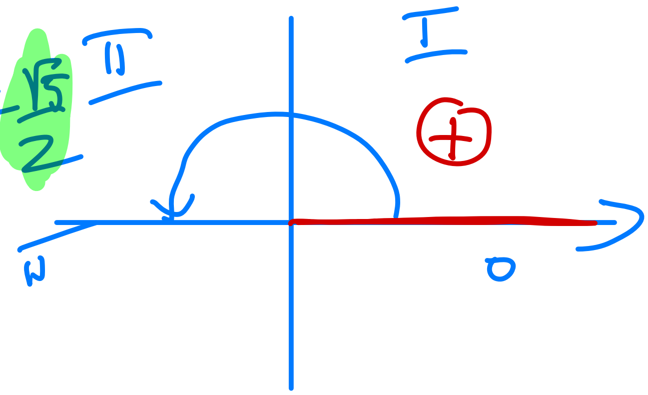
When we compose trig functions with other trig inverses (i.e. tangent composed with arccosine) we can make use of right triangles to find exact values of compositions that involve inverse trig functions.

Example 3. *Find the exact value.*

a. $\tan\left(\arccos \frac{2}{3}\right)$

b. $\cos\left[\arcsin\left(-\frac{3}{5}\right)\right]$

a) $\tan(\underbrace{\arccos(\tfrac{2}{3})}_{u}) = \tan u = \frac{\sqrt{5}}{2}$



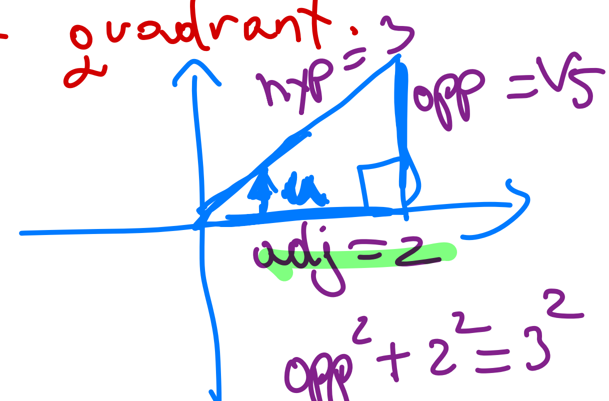
$\arccos\left(\frac{2}{3}\right) = u$

$R: [0, \pi]$

$\boxed{\cos u = \frac{2}{3}}$

positive so u is an angle in first quadrant.

$\frac{adj}{hyp} = \frac{2}{3}$



$opp^2 + 2^2 = 3^2$

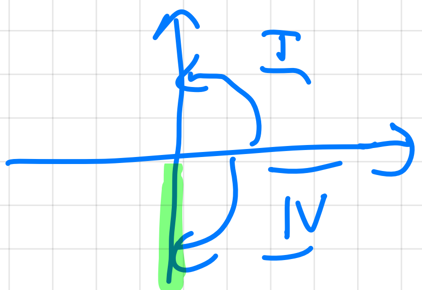
$opp = \sqrt{5}$

b) $\cos\left(\underbrace{\arcsin\left(-\frac{3}{5}\right)}_{u}\right) = \cos u = \frac{adj}{hyp} = \frac{4}{5}$

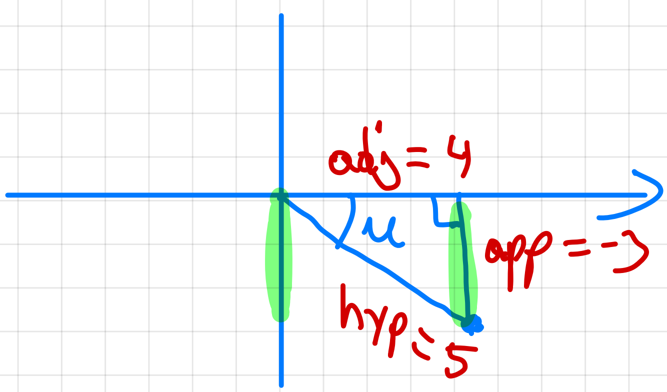
- $\arcsin\left(-\frac{3}{5}\right) = u$

$\boxed{\sin u = -\frac{3}{5}} \Longrightarrow \frac{opp}{hyp} = \frac{-3}{5}$

Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



u is in quadrant IV



$adj^2 + (-3)^2 = 5^2$

$adj = \sqrt{16}$

Example 4. *Write each of the following as an algebraic expression in x.*

a. $\sin(\arccos 3x), \quad 0 \le x \le \frac{1}{3}$

b. $\cot(\arccos 3x), \quad 0 \le x < \frac{1}{3}$

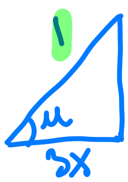
a) $\sin(\underbrace{\arccos 3x}_{u}) = \sin u = \frac{opp}{hyp} = \sqrt{1-9x^2}$

$\int \sqrt{1-9x^2}\, dx = ?$

$\arccos(3x) = u$

$\cos u = 3x$

$\frac{adj}{hyp} = \frac{3x}{1}$



$opp = \sqrt{1-9x^2}$

$opp^2 + (3x)^2 = 1$

$opp^2 = 1-(3x)^2$

$opp^2 = 1-9x^2$

$opp = \sqrt{1-9x^2}$

b) $\cot(\underbrace{\arccos 3x}_{u}) = \cot u = \frac{adj}{opp} = \int \frac{3x}{\sqrt{1-9x^2}}\, dx$