1. Factoring out the Greatest Common Factor (GCF)

Key Concept: The first step in factoring any expression is to check whether all terms share a common factor.

Process:

1. Identify the GCF of all terms in the expression
2. Factor out the GCF from each term
3. Write as GCF $\times$ (remaining factors)

Example 1: Factor $12x^5 + 18x^4 - 24x^3$

12 : 1, 2, 3, 4, 6, 12
18 : 1, 2, 3, 6, 9, 18
24 : 1, 2, 3, 4, 6, 8, 12, 24

(a) Find the GCF of the terms.

- GCF of coefficients: GCF of 12, 18, and 24 = 6
- GCF of variables: GCF of x^5, x^4, and x^3 = x^3
- Therefore, the GCF is $6x^3$

(b) Factor out the GCF from each term.
$12x^5 + 18x^4 - 24x^3 = 6x^3\left(\frac{12x^5}{6x^3} + \frac{18x^4}{6x^3} - \frac{24x^3}{6x^3}\right)$

$= 6x^3(2x^2 + 3x - 4)$

Example 2: Factor $3y(2y - 5) + (2y - 5)$

(a) Recognize that $(2y - 5)$ appears in both terms. Therefore, the GCF is $2y-5$

(b) Factor out the GCF from each term.
$3y(2y - 5) + (2y - 5) = (2y - 5)(3y + 1)$

$= (2y-5)(3y+1)$

check

Example 3: Factor $-4x^3 - 12x^2 + 8x$ $= -4x\left(x^2 + 3x - 2\right)$

$$\frac{-4x^3}{-4x} = x^2 \qquad \frac{-12x^2}{-4x} = 3x \qquad \frac{8x}{-4x} = -2$$

2. Factoring by Grouping

Key Concept: When an expression has four terms, we can sometimes factor it by grouping terms in pairs.

Process:

1. Group the first two terms and last two terms (or different order)
2. Factor out the GCF from each group
3. If the expressions in parentheses are identical, factor out this common binomial

Example 1: Factor $2ax - 6ay + 5x - 15y$

$$\frac{2ax}{2a} = x \qquad \frac{-6ay}{2a} = -3y$$

$= (2ax - 6ay) + (5x - 15y)$

$= 2a(x - 3y) + 5(x - 3y)$

$= (x - 3y) \cdot (2a + 5)$

Example 2: Factor $m^2 - 3n + 3m - mn$

$= (m^2 + 3m) - (3n + mn)$

$= m(m + 3) - n(3 + m)$

$= (3 + m)(m - n)$

$8 = 4 \cdot 2$

$ax^2+bx+c = (x-p)(x-q)$

Key Concept: A trinomial in the form $ax^2 + bx + c$ can often be factored into two binomials. $= (x-p)(x-q)$

A. Factoring Trinomials with Leading Coefficient 1: $x^2 + bx + c$

$a=1$

Process:

1. Find two numbers that multiply to give c and add to give b
2. Use these numbers to write the factored form: $(x + p)(x + q)$ where $p \times q = c$ and $p + q = b$

Example: Factor $x^2 + 7x + 12$

$a=1$ $b=7$ $c=12$

factors of $c=12$

1, 2, 3, 4, 6, 12

Find p and q s.t $p \cdot q = 12$

$p+q=7$

$p=3$, $q=4$ ✓

$$\boxed{x^2+7x+12 = (x+3)(x+4)}$$

B. Factoring Trinomials with Leading Coefficient $\neq 1$: $ax^2 + bx + c$

Process (AC Method):

1. Multiply $a \times c$
2. Find two numbers that multiply to give $a \times c$ and add to give b
3. Rewrite the middle term using these two numbers
4. Factor by grouping

Example: Factor $3x^2 + 14x + 15$ $= \boxed{(3x+5)(x+3)}$

$a=3$ $b=14$ $c=15$

(a) Multiply $a \times c = 3 \cdot 15 = 45$

(b) Find two numbers that multiply to give 45 and add to give 14.
Factors of 45: Check sums:

1, 5, 15, 9, 45

$5 \cdot 9 = 45$

$5+9=14$

(c) Rewrite the middle term using 5 and 9. $=14$

$3x^2 + 14x + 15 = 3x^2 + 5x + 9x + 15$

(d) Factor by grouping.

$3x^2 + 5x + 9x + 15 =$ $x(3x+5) + 3(3x+5) = \boxed{(3x+5)(x+3)}$

C. Factoring Perfect Square Trinomials

Perfect Square Trinomial Formulas:

$$a^2 + 2ab + b^2 = (a+b)^2$$
$$a^2 - 2ab + b^2 = (a-b)^2$$

caution:

$(x+2)^2 \neq x^2 + 2^2$

$\cancel{\neq} \; x^2 + 4$

$(x+2)^2 = (x+2)(x+2)$

$= x^2 + 2x + 2x + 4$

$= x^2 + 4x + 4$

Example 1: Factor $4x^2 - 28x + 49$ $= (2x-7)^2$

$(2x)^2$ $\quad 7^2$

$a = 2x$ $\quad b = 7$

$a = 2x$

$b = 7$

$a^2 - 2ab + b^2 = (2x)^2 - 2(2x)(7) + 7^2$

$= \boxed{4x^2 - 28x + 49}$

$4x^2 - 28x + 49 = (2x-7)^2$

Example 2: Factor $81c^4 + 90c^2d + 25d^2$

check $2(9c^2)(5d) = 90c^2d$

$(9c^2)^2$ $\quad (5d)^2$

yes perfect square trin. $\quad (a+b)^2 = (9c^2 + 5d)^2$

$81c^4 + 90c^2d + 25d^2 = (9c^2 + 5d)^2$

4. Factoring Difference of Squares

Key Concept: A binomial in the form $a^2 - b^2$ can be factored into $(a+b)(a-b)$.

Difference of Squares Formula:

$\square^2 - \triangle^2 = (\square - \triangle)(\square - \triangle)$

$$\boxed{a^2 - b^2 = (a+b)(a-b)}$$

Example 1: Factor $x^2 - 25$

$\boxed{x}^2 - 5^2 = (x-5)(x+5)$

$a^2 - b^2 = (a-b)(a+b)$

Example 2: Factor $16y^2 - 9z^2$ $= (\boxed{4y})^2 - (3z)^2$

$16y^2 - 9z^2 = (4y)^2 - (3z)^2 = (4y-3z)(4y+3z)$

5. Factoring Sum and Difference of Cubes

Key Concept: Binomials in the form $a^3 \pm b^3$ have specific factoring formulas.

Sum and Difference of Cubes Formulas:

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Example 1: Factor $x^3 - 27$ $= x^3 - 3^3 = (x-3)(x^2+3x+9)$

$a = x$
$b = 3$

Example 2: Factor $8y^3 + 125$ $= (2y)^3 + 5^3 = (2y+5)(4y^2 - 10y + 25)$

$a = 2y$ $b = 5$

6. Factoring Using Substitution

Key Concept: When an expression contains a repeating pattern, substitution can simplify the factoring process.

Process:

1. Identify the repeating pattern and substitute it with a simpler variable
2. Factor the resulting expression
3. Substitute back the original expression

Example: Factor $(x^2-5)^2 + 2(x^2-5) - 24$ $= (x^2+1)(x^2-9)$

$= (x^2+1)(x-3)(x+3)$

1) let $x^2-5 = u$

$\underbrace{u^2 + 2u - 24}$

2) factor p, q ? s.t $p \cdot q = -24$, $p+q = 2$

$\pm$! $-24: 1, 2, 3, 4, 6, 8, 12, 24$ choose $-4, 6$

$(u+6)(u-4) \overset{\text{check}}{=} (u^2-4u+6u-24)$

3) $(\underbrace{x^2-5}_{u}+6)(\underbrace{x^2-5}_{u}-4) = (x^2+1)(x^2-9)$

7. Factoring with Negative and Rational Exponents

Key Concept: When factoring expressions with negative or rational exponents, we can still use our standard factoring techniques after identifying the GCF.

Example 1: Factor $x^{-6} + 5x^{-5} + 7x^{-4}$

$= x^{-6}\left(\frac{x^{-6}}{x^{-6}} + \frac{5x^{-5}}{x^{-6}} + \frac{7x^{-4}}{x^{-6}}\right)$

$-6 \quad -5 \quad -4 \quad 0 \quad 1 \quad 4$

$= x^{-6}\left(1 + 5x^{-5-(-6)} + 7x^{-4-(-6)}\right)$ $\qquad \frac{x^a}{x^b} = x^{a-b}$

$= x^{-6}\left(1 + 5x^1 + 7x^2\right)$ $= x^{-6}(1+5x+7x^2)$

Example 2: Factor $x(2x+5)^{-1/2} + (2x+5)^{1/2}$

$$= (2x+5)^{-1/2}\left[\frac{x(2x+5)^{-1/2}}{(2x+5)^{-1/2}} + \frac{(2x+5)^{1/2}}{(2x+5)^{-1/2}}\right]$$

$-1/2 \quad 0 \quad 1/2$

$$= (2x+5)^{-1/2}\left[x + (2x+5)^{1/2-(-1/2)}\right] = (2x+5)^{-1/2}\left[x + (2x+5)^{1}\right]$$

Recall $x^{-a} = \frac{1}{x^{a}}$

$$= \frac{1}{(2x+5)^{1/2}}\left[x + (2x+5)\right]$$

Factoring Summary

Factoring Technique	**Example**
1. GCF	$6x^3y + 12x^2y^2 = 6x^2y(x+2y)$
2. Factoring by Grouping	$ax + ay + bx + by = a(x+y) + b(x+y) = (x+y)(a+b)$
3. Trinomials with Leading Coefficient 1	$x^2 + 5x + 6 = (x+2)(x+3)$
4. Trinomials with Leading Coefficient $\neq 1$	$2x^2 + 7x + 3 = (2x+1)(x+3)$
5. Perfect Square Trinomials	$x^2 + 6x + 9 = (x+3)^2$
6. Difference of Squares	$x^2 - 9 = (x+3)(x-3)$
7. Sum of Cubes	$x^3 + 8 = (x+2)(x^2 - 2x + 4)$
8. Difference of Cubes	$x^3 - 8 = (x-2)(x^2 + 2x + 4)$
9. Substitution	$(x^2+1)^2 - 4(x^2+1) = (x^2+1-4)(x^2+1+1)$
10. Negative/Rational Exponents	$x^{-2} + 3x^{-1} = x^{-2}(1+3x)$

Remember to always look for the GCF first!