Monday March 31st

HW 12 due Today

HW 13 due Wed

Exam 3 is on Wed April 9th
on Chapter 3 (Exp & Logs)
Chapter 9

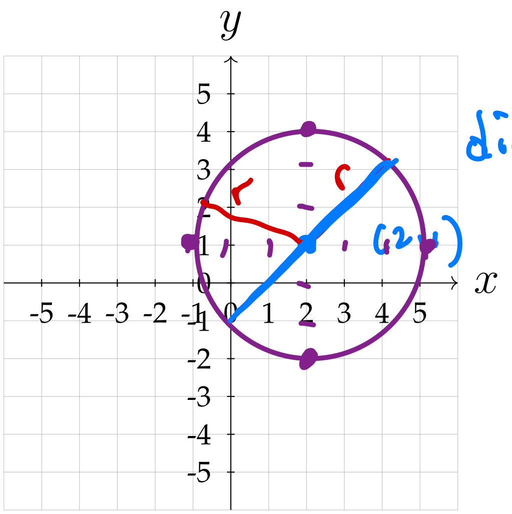
Day 15

9.1 The Circles and 9.2 The Ellipse

1. Circles

Next, we will start graphing specific categories of equations such as circles, ellipses, hyperbolas and parabola. We begin with circles.

Definition: A circle is the set of all points in a plane that are equidistant from a fixed point called center. The fixed distance from any point on the circle to the center is called the radius.



diameter = 2r

center : (h, k)

y coord of center

x coord. of center

(h, K) = (2, 1)

r = 3

Standard form of an equation of a circle

Given a circle centered at (h, k) with radius r, the standard form is

$$\boxed{(x-h)^2 + (y-k)^2 = r^2}$$

$(x-h)^2+(y-k)^2=r^2$

Example 1

1. $(x-4)^2+(y+3)^2=25$

$(x-4)^2+(y-(-3))^2=5^2$

$\boxed{(h,k)=(4,-3) \quad r=5}$

2. $x^2+(y-1/2)^2=12$

$(x-0)^2+(y-1/2)^2=$

$(h,k)=(0,1/2)$

$r=\sqrt{12}=\sqrt{4\cdot 3}=\sqrt{4}\sqrt{3}=2\sqrt{3}$

3. $x^2+y^2=7$

$(x-0)^2+(y-0)^2=(\sqrt{7})^2$

$(h,k)=(0,0)$

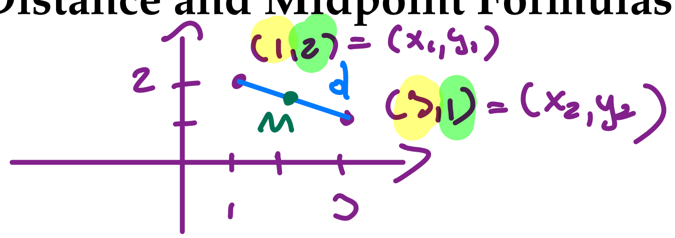
$r=\sqrt{7}$

Example 2 Write the standard form of an equation of a circle with center $(\underset{h}{-4},\underset{k}{6})$ and radius 2.

$(x-h)^2+(y-k)^2=r^2$

$(x+4)^2+(y-6)^2=2^2$

2. Recall: Distance and Midpoint Formulas



Distance Formula: The distance d between two points (x_1, y_1) and (x_2, y_2) is:

$d=\sqrt{(3-1)^2+(1-2)^2}=\sqrt{4+1}=\sqrt{5}$

$$\boxed{d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}$$

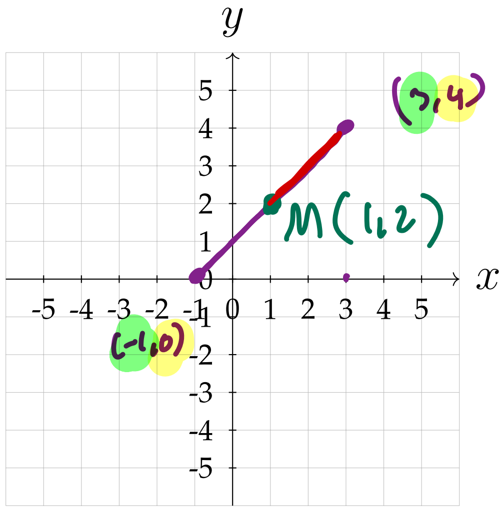
Midpoint Formula: The midpoint $M(x, y)$ of a line segment with endpoints (x_1, y_1) and (x_2, y_2) is:

$$\boxed{M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)}$$

$M\left(\frac{3+1}{2}, \frac{1+2}{2}\right)=(2, 3/2)=(2, 1.5)$

$(x-h)^2+(y-k)^2=r^2$

Example 3 Write the standard form of an equation of a circle with the endpoints of the diameter as $(-1,0)$ and $(3,4)$. Graph the circle.



$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{3+(-1)}{2}, \frac{0+4}{2}\right)$

midpoint $=(1,2)=$ center

$(h,k)=(1,2)$

$r=\sqrt{(3-1)^2+(4-2)^2}=\sqrt{4+4}=\sqrt{8}=\sqrt{4\cdot 2}$
$=2\sqrt{2}$

Answer: $\boxed{(x-1)^2+(y-2)^2=8}$

General form of an Equation of a circle

An equation of a circle written in the form $x^2+y^2+Ax+By+C=0$ is called the general form of an equation of a circle.

Example 4 Write the equation of the circle in standard form by **completing the square.**

$x^2+y^2+10x-6y+25=0$

① group the x term together and group the y terms together.

$x^2+10x+25+y^2-6y+9=-25+25+9$

② complete the square in x and in y.

$\boxed{(x+5)^2+(y-3)^2=9}$

center: $(-5,3)$

r: $r=3$

completing the square

$x^2+10x+25=-25+25$

take half of coefficient of x, square and add it to both side.

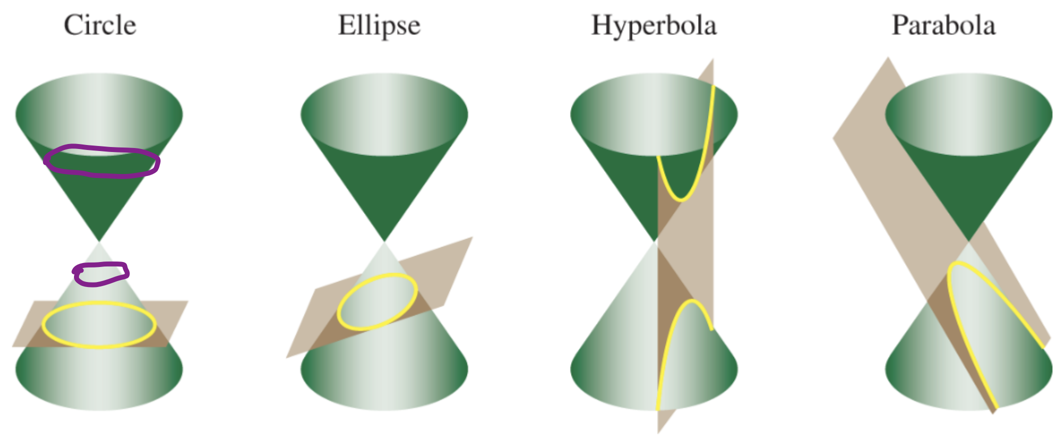
$\left(\frac{10}{2}\right)^2=25$

goal

$(x+5)^2=0$

2. Ellipses

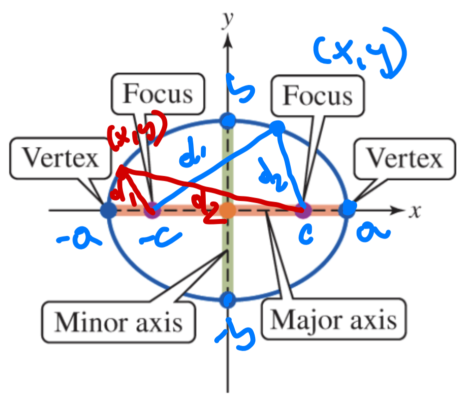
In this section we study ellipses from a geometric point of view.



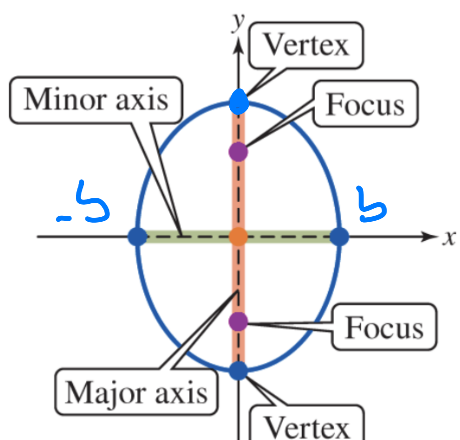
Definition: An ellipse is the set of all points in the plane such that the sum of the distance between a point (x, y) and two fixed points is a constant.

The fixed points are called foci (plural of focus) of the ellipse.

An ellipse may be elongated in any direction. We will study only those that are elongated horizontally and vertically.



- center: (0,0)
- elongated horizontally (along the x axis)
- foci (c,0) & (-c,0)
- vertices (a,0) & (-a,0)
- covertices (0,b) & (0,-b)



- center (0,0)
- elongated vertically (along y axis)
- foci (0,c) & (0,-c)
- vertices: (0,a) & (0,-a)
- covertices: (b,0) & (-b,0)

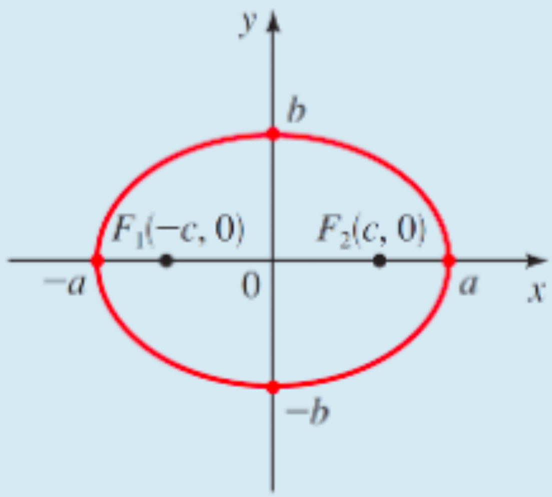
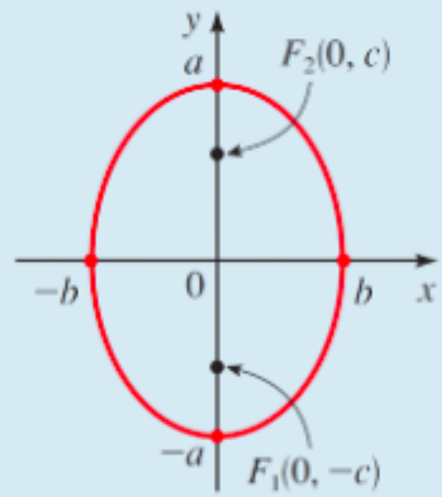
Length of major axis: 2a
Length of minor axis: 2b

3. Equations and Graphs of ellipses

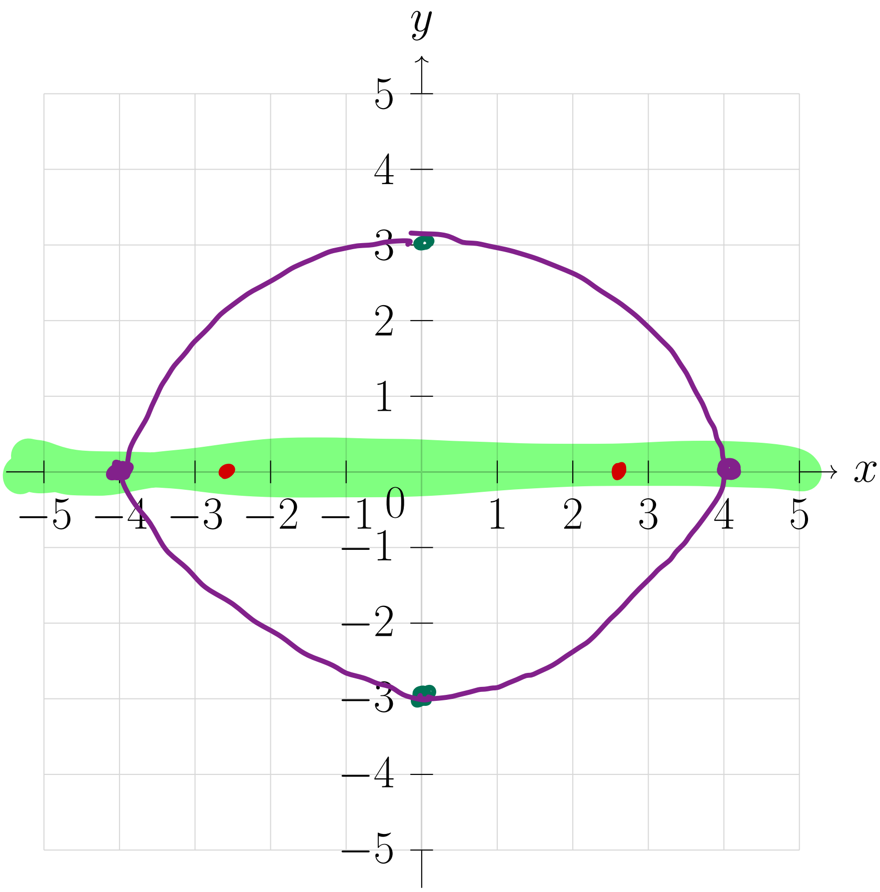
The following box summarizes the equation and features of an ellipse centered at the origin.

ELLIPSE WITH CENTER AT THE ORIGIN

The graph of each of the following equations is an ellipse with center at the origin and having the given properties.

EQUATION	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $a > b > 0$	$\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$ $a > b > 0$
VERTICES	$(\pm a, 0)$	$(0, \pm a)$
MAJOR AXIS	Horizontal, length $2a$	Vertical, length $2a$
MINOR AXIS	Vertical, length $2b$	Horizontal, length $2b$
FOCI	$(\pm c, 0)$, $c^2 = a^2 - b^2$	$(0, \pm c)$, $c^2 = a^2 - b^2$
GRAPH		

Example 5: Given $\frac{x^2}{16} + \frac{y^2}{9} = 1$, graph the ellipse and identify the center, foci and vertices.



$$\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$$

- since $4 > 3$ we have ellips elongated along xaxis
- vertices $a = 4$ $(4,0)$ & $(-4,0)$
- covertices $b = 3$ $(0,-3)$ & $(0,3)$

$c^2 = a^2 - b^2$

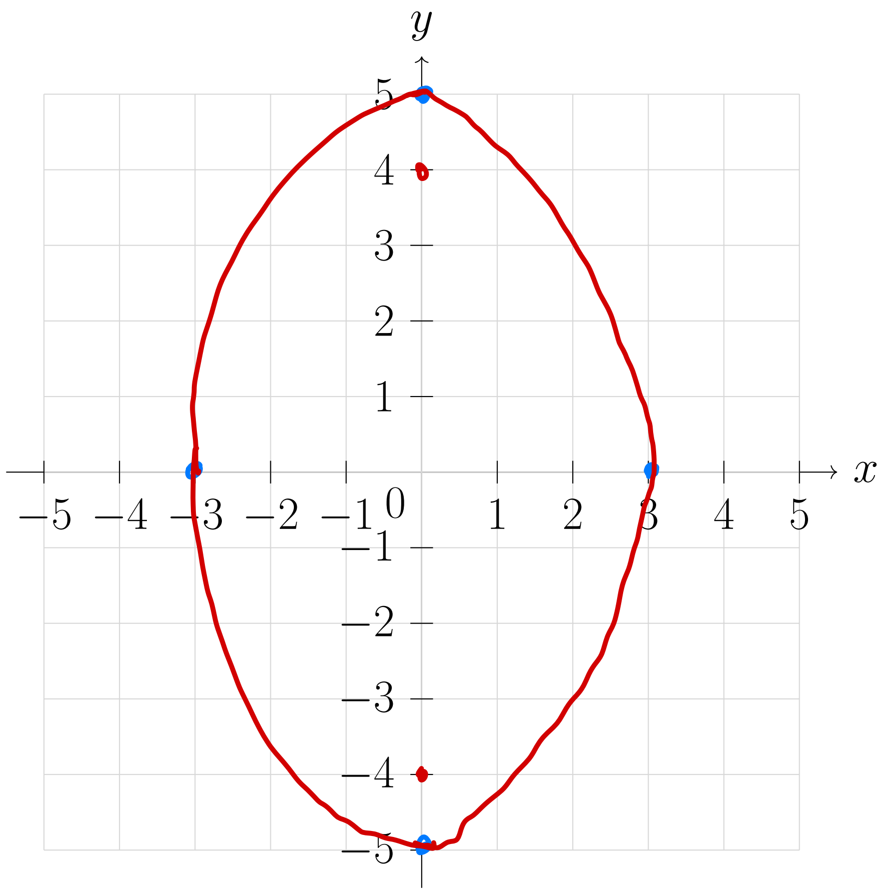
- foci $c^2 = 4^2 - 3^2$
  $= 16 - 9$
  $= 7$
  $(\sqrt{7}, 0)$ & $(-\sqrt{7}, 0)$
  $(2.6, 0)$ & $(-2.6, 0)$

Major axis: $2a = 2 \cdot 4 = 8$

minor axis: $2b = 6$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

Example 6: Find the foci of the ellipse $\frac{25x^2 + 9y^2}{225} = \frac{225}{225}$, and sketch its graph.



$\frac{x^2}{9} + \frac{y^2}{25} = 1$

$\frac{x^2}{3^2} + \frac{y^2}{5^2} = 1$

$b = 3 \quad a = 5$

$c^2 = a^2 - b^2$
$= 25 - 9$
$= 16$
$c = 4$

Calculator Instructions:

First solve for y:

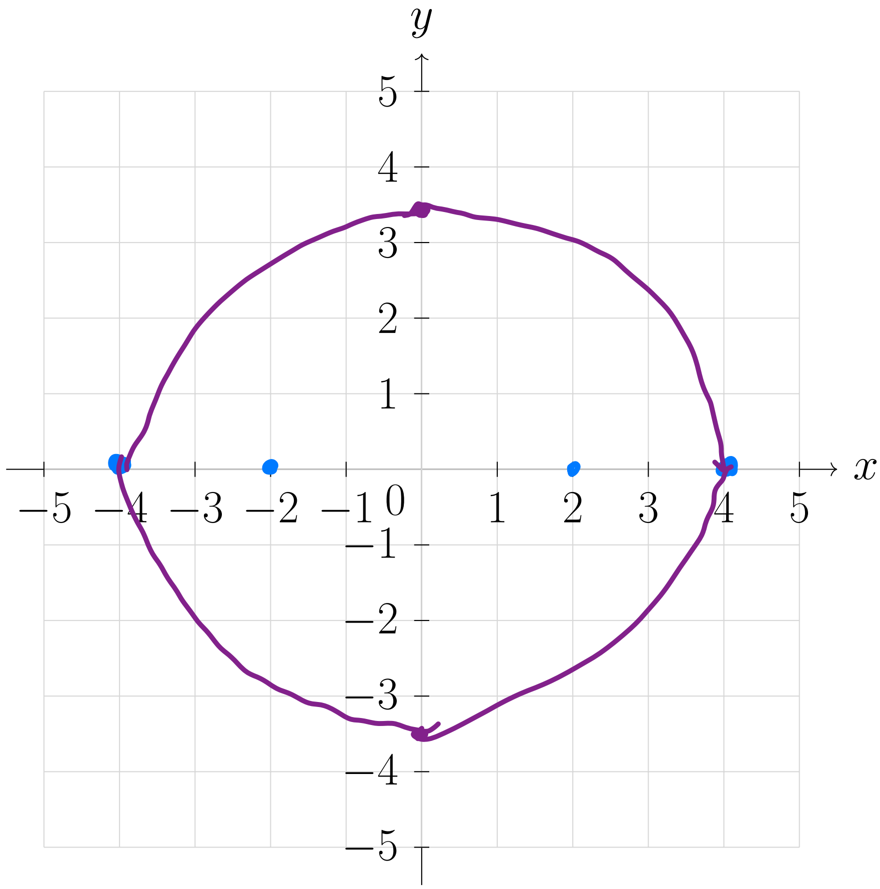
$25x^2 + 9y^2 = 225$

$\frac{9y^2}{9} = \frac{225 - 25x^2}{9}$

$y^2 = \frac{225 - 25x^2}{9}$

$y = \pm\sqrt{\frac{225 - 25x^2}{9}}$

Example 7: The vertices of an ellipse are $(\pm 4, 0)$, and the foci are $(\pm 2, 0)$. Find its equation, and sketch the graph.



$a = 4$

$c = 2$

$c^2 = a^2 - b^2$

$4 = 16 - b^2$

$-12 = -b^2$

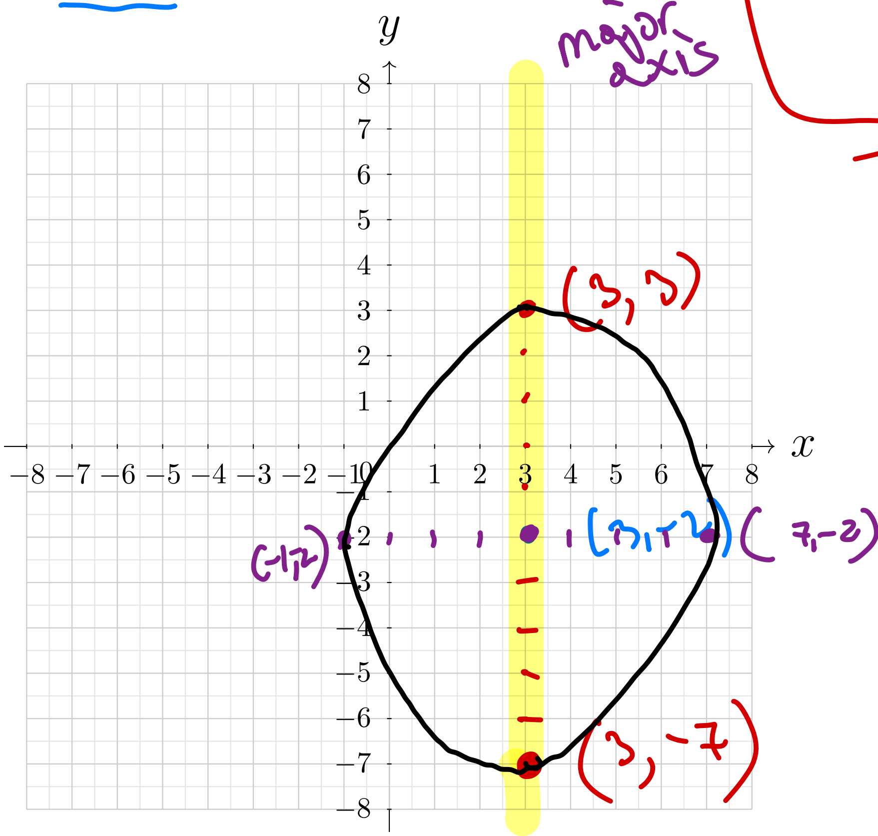
$b^2 = 12$

$b = 2\sqrt{3} \approx 3.4$

4. Graphing and Ellipse centered at (h, k)

Replacing x by $x - h$ and y by $y - k$ shifts the graph of an ellipse h units horizontally and k units vertically.

Example 8: Given $\dfrac{(x-3)^2}{16} + \dfrac{(y+2)^2}{25} = 1$, graph the ellipse and identify the center, foci and vertices.

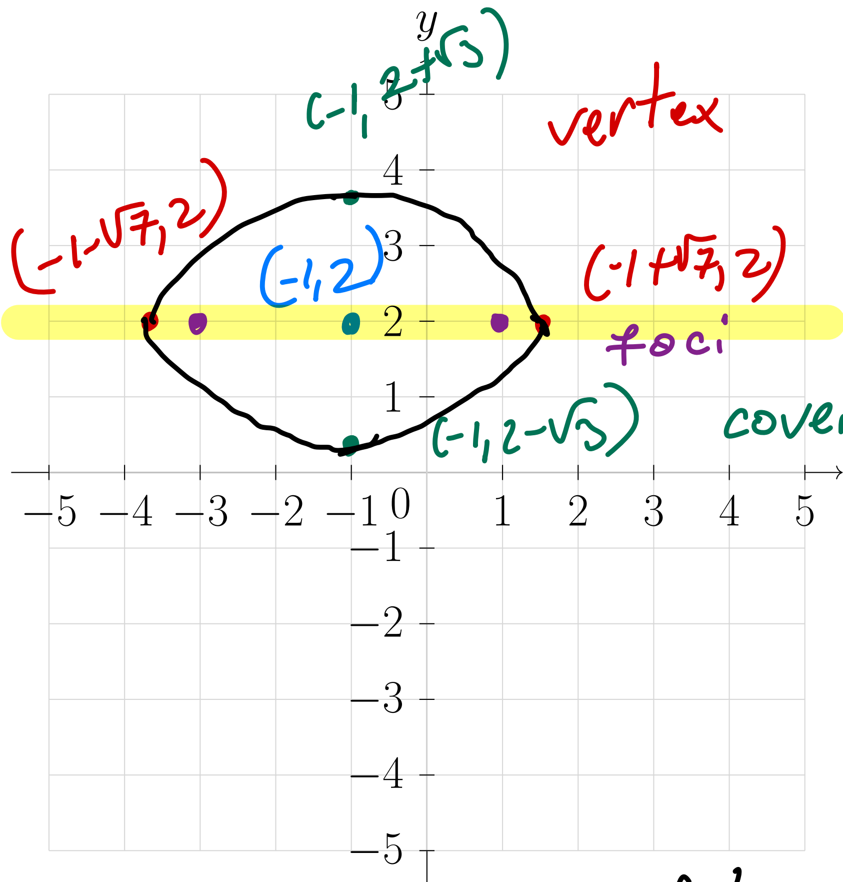


center: (3,-2) ✓
elongating along y axis
a=5
b=4

vertices: (3,3) & (3,-7)

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

Example 9: Practice: Given $3x^2 + 7y^2 + 6x - 28y + 10 = 0$, write the equation of the ellipse in standard form, graph the ellipse and identify the center, foci and vertices.



$3x^2+7y^2+6x-28y=-10$

$3x^2+6x+7y^2-28y=-10$

$3(x^2+2x\quad)+7(y^2-4y\quad)=-10$

$3(x^2+2x\quad)+7(y^2-4y\quad)=-10$

$3(x^2+2x+1)+7(y^2-4y+4)=-10+3+28$

$\frac{3(x+1)^2+7(y-2)^2}{21}=\frac{21}{21}$

$$\boxed{\frac{(x+1)^2}{7}+\frac{(y-2)^2}{3}=1}$$

$a=\sqrt{7}$, $b=\sqrt{3}$ (elongated horizontally)

center $(-1, 2)$

foci: $c^2=a^2-b^2=7-3=4$ $\boxed{c=2}$