Guided Notes
7.3 Systems of Nonlinear Equations

1. Introduction to Systems of Nonlinear Equations

A **system of nonlinear equations** is a system of two or more equations in two or more variables containing at least one equation that is not linear.

Recall that a linear equation can take the form $Ax + By + C = 0$. Any equation that cannot be written in this form is nonlinear.

The methods for solving systems of nonlinear equations are similar to those for linear systems:

- Substitution Method
- Elimination Method

2. Solving Using Substitution

The substitution method involves:

1. Solve one equation for one variable
2. Substitute the result into the second equation
3. Solve for the remaining variable
4. Check your solutions in both equations

Intersection of a Parabola and a Line

There are three possible types of solutions for a system involving a parabola and a line:

- **No solution:** The line will never intersect the parabola
- **One solution:** The line is tangent to the parabola and intersects at exactly one point
- **Two solutions:** The line crosses the parabola and intersects at two points

Example 1: Solve the system of equations.

$$\begin{aligned} x - y &= -1 \\ y &= x^2 + 1 \end{aligned}$$

Solution:
From the first equation, solve for x:

$$x = y - 1$$

Substitute into the second equation:

$$\begin{aligned} y &= (y-1)^2 + 1 \\ y &= y^2 - 2y + 1 + 1 \\ y &= y^2 - 2y + 2 \\ 0 &= y^2 - 3y + 2 \\ 0 &= (y-1)(y-2) \end{aligned}$$

So $y = 1$ or $y = 2$.
When $y = 1$: $x = 1 - 1 = 0$
When $y = 2$: $x = 2 - 1 = 1$
Solutions: $(0, 1)$ and $(1, 2)$

Intersection of a Circle and a Line

Just as with a parabola and a line, there are three possible outcomes when solving a system of equations representing a circle and a line:

- **No solution:** The line does not intersect the circle
- **One solution:** The line is tangent to the circle and intersects at exactly one point
- **Two solutions:** The line crosses the circle and intersects at two points

Example 2: Find the intersection of the given circle and line by substitution.

$$x^2 + y^2 = 25$$
$$x - y = 1$$

Solution:
From the second equation: $x = y + 1$
Substitute into the first equation:

$$\begin{aligned}
(y+1)^2 + y^2 &= 25 \\
y^2 + 2y + 1 + y^2 &= 25 \\
2y^2 + 2y + 1 &= 25 \\
2y^2 + 2y - 24 &= 0 \\
y^2 + y - 12 &= 0 \\
(y+4)(y-3) &= 0
\end{aligned}$$

So $y = -4$ or $y = 3$.
When $y = -4$: $x = -4 + 1 = -3$
When $y = 3$: $x = 3 + 1 = 4$
Solutions: $(-3, -4)$ and $(4, 3)$

3. Solving Using Elimination

When both equations in the system have like variables of the second degree, solving them using elimination by addition is often easier than substitution.

The elimination method is particularly useful for systems involving:

- Two circles
- A circle and an ellipse
- Two ellipses

Intersection of a Circle and an Ellipse

There are five possible types of solutions when solving a system of equations representing a circle and an ellipse:

- **No solution:** The circle and ellipse do not intersect
- **One solution:** The circle and ellipse are tangent to each other
- **Two solutions:** The circle and ellipse intersect at two points
- **Three solutions:** The circle and ellipse intersect at three points
- **Four solutions:** The circle and ellipse intersect at four points

Example 3: Solve the system of nonlinear equations.

$$\begin{aligned} x^2 + y^2 &= 26 \\ 3x^2 + 25y^2 &= 100 \end{aligned}$$

Solution:
Multiply the first equation by -3:

$$-3x^2 - 3y^2 = -78$$

Add to the second equation:

$$\begin{aligned} 3x^2 + 25y^2 &= 100 \\ -3x^2 - 3y^2 &= -78 \\ \hline 22y^2 &= 22 \\ y^2 &= 1 \\ y &= \pm 1 \end{aligned}$$

When $y = 1$: $x^2 + 1 = 26 \Rightarrow x^2 = 25 \Rightarrow x = \pm 5$
When $y = -1$: $x^2 + 1 = 26 \Rightarrow x^2 = 25 \Rightarrow x = \pm 5$
Solutions: $(5, 1)$, $(-5, 1)$, $(5, -1)$, and $(-5, -1)$

Practice Problems

Solve each system of nonlinear equations using substitution or elimination.

Problem 1:

$$\begin{aligned} y &= x + 1 \\ x^2 + y^2 &= 25 \end{aligned}$$

Solution:
Substitute $y = x + 1$ into the second equation:

$$\begin{aligned} x^2 + (x+1)^2 &= 25 \\ x^2 + x^2 + 2x + 1 &= 25 \\ 2x^2 + 2x + 1 &= 25 \\ 2x^2 + 2x - 24 &= 0 \\ x^2 + x - 12 &= 0 \\ (x+4)(x-3) &= 0 \end{aligned}$$

So $x = -4$ or $x = 3$.
When $x = -4$: $y = -4 + 1 = -3$
When $x = 3$: $y = 3 + 1 = 4$
Solutions: $(-4, -3)$ and $(3, 4)$

Problem 2:

$$\begin{aligned} y &= -x \\ x^2 + y^2 &= 9 \end{aligned}$$

Solution:
Substitute $y = -x$ into the second equation:

$$\begin{aligned} x^2 + (-x)^2 &= 9 \\ x^2 + x^2 &= 9 \\ 2x^2 &= 9 \\ x^2 &= \frac{9}{2} \\ x &= \pm\frac{3}{\sqrt{2}} = \pm\frac{3\sqrt{2}}{2} \end{aligned}$$

When $x = \frac{3\sqrt{2}}{2}$: $y = -\frac{3\sqrt{2}}{2}$
When $x = -\frac{3\sqrt{2}}{2}$: $y = \frac{3\sqrt{2}}{2}$
Solutions: $\left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$ and $\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$

Problem 3:

$$x^2 + y^2 = 25$$
$$x^2 - y^2 = 1$$

Solution:
Add the two equations:

$$\begin{aligned} x^2 + y^2 &= 25 \\ x^2 - y^2 &= 1 \\ \hline 2x^2 &= 26 \\ x^2 &= 13 \\ x &= \pm\sqrt{13} \end{aligned}$$

From the second equation: $y^2 = x^2 - 1 = 13 - 1 = 12$
So $y = \pm 2\sqrt{3}$
Solutions: $(\sqrt{13}, 2\sqrt{3})$, $(\sqrt{13}, -2\sqrt{3})$, $(-\sqrt{13}, 2\sqrt{3})$, and $(-\sqrt{13}, -2\sqrt{3})$

Problem 4:

$$2x^2 + 4y^2 = 4$$
$$x^2 + y^2 = 4$$

Solution:
Multiply the second equation by -2:

$$-2x^2 - 2y^2 = -8$$

Add to the first equation:

$$\begin{aligned} 2x^2 + 4y^2 &= 4 \\ -2x^2 - 2y^2 &= -8 \\ \hline 2y^2 &= -4 \\ y^2 &= -2 \end{aligned}$$

Since $y^2 = -2$ has no real solutions, this system has **no real solutions**.
(The system would have complex solutions: $y = \pm i\sqrt{2}$, but typically we only consider real solutions in this context.)