

1. Adding, subtracting, multiplying and dividing functions

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg , and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers.

ALGEBRA OF FUNCTIONS

Let f and g be functions with domains A and B . Then the functions $f + g$, $f - g$, fg , and f/g are defined as follows.

$$(f + g)(x) = f(x) + g(x) \quad \text{Domain } A \cap B$$

$$(f - g)(x) = f(x) - g(x) \quad \text{Domain } A \cap B$$

$$(fg)(x) = f(x)g(x) \quad \text{Domain } A \cap B$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \text{Domain } \{x \in A \cap B \mid g(x) \neq 0\}$$

Practice: Perform the following function operations for f and g and find the domains of the resulting functions.

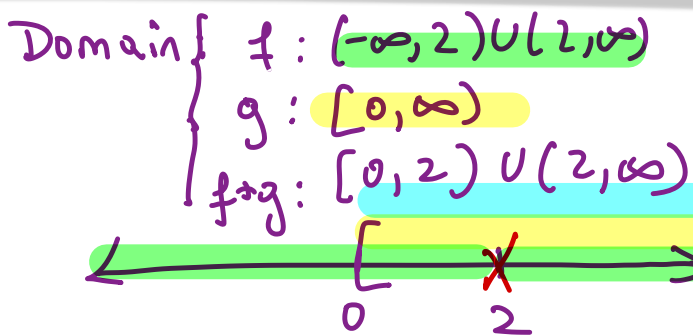
$$f(x) = \frac{1}{x-2} \text{ and } g(x) = \sqrt{x}.$$

$$(a) (f + g)(x) = \frac{1}{x-2} + \sqrt{x}$$

$$(b) (f - g)(x) = \frac{1}{x-2} - \sqrt{x}$$

$$(c) (fg)(x) = f(x)g(x) = \frac{1}{x-2} \cdot \sqrt{x} = \frac{\sqrt{x}}{x-2} \quad \text{domain } fg: [0, 2) \cup (2, \infty)$$

$$(d) \left(\frac{f}{g}\right)(x) = \frac{\frac{1}{x-2}}{\sqrt{x}} = \frac{1}{\sqrt{x}(x-2)} \quad \text{domain: } (0, 2) \cup (2, \infty)$$



2. Composition of Functions

Another way to combine two functions f and g is to **compose them**. Composing the function $f(x)$ with the function $g(x)$, denoted $(f \circ g)(x)$, means that the output of g becomes the input of f . In other words $(f \circ g)(x) = \underline{f(g(x))}$ or the function f is evaluated at $g(x)$.

Example: Let $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{4}{x-1}$. Find the following:

(a) $f \circ g$ and its domain

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) = \frac{1}{g(x)+2} = \frac{1}{\frac{4}{x-1} + 2} = \frac{1}{\frac{4 + 2x - 2}{x-1}} \\ &= \frac{1}{\frac{2+2x}{x-1}} = 1 \cdot \frac{x-1}{2x+2} \end{aligned}$$

Domain: $x \neq -1$ $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
 $x \neq 1$

(b) $g \circ f$ and its domain

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) = \frac{4}{f(x)-1} = \frac{4}{\frac{1}{x+2} - 1} = \frac{4}{\frac{1 - (x+2)}{x+2}} = \frac{4}{\frac{1-x-2}{x+2}} \\ &= \frac{4}{\frac{-x-1}{x+2}} = 4 \cdot \frac{x+2}{-x-1} \end{aligned}$$

Domain: $x \neq -1$, $x \neq -2$
 $(-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$

NOTE The domain of $f \circ g$ is the intersection of the domain of inner function g and the resulting function $f \circ g$.

Practice: Exercise 1

Let $f(x) = \frac{x}{x+3}$ and $g(x) = 8x - 3$. Find the following

1. $(f \circ g)(x)$ and its domain

$$(f \circ g)(x) = f(g(x)) = \frac{g(x)}{g(x)+3} = \frac{8x-3}{8x-3+3} = \frac{8x-3}{8x}$$

$$\text{Domain: } x \neq 0 \text{ so: } (-\infty, 0) \cup (0, \infty)$$

2. $(g \circ f)(x)$ and its domain

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) = 8f(x) - 3 = 8\left(\frac{x}{x+3}\right) - 3 = \frac{8x}{x+3} - 3 \frac{x+3}{x+3} \\ &= \frac{8x - 3(x+3)}{x+3} = \frac{5x-9}{x+3}; \quad \text{Domain: } x \neq -3 \\ &\quad (-\infty, -3) \cup (-3, \infty) \end{aligned}$$

Practice: Exercise 2

Let $f(x) = x^3 + 8$ and $g(x) = x - 5$ and $h(x) = \sqrt{x}$. Find $f \circ g \circ h$.

$$\begin{aligned} (f \circ g \circ h)(x) &= f(g(h(x))) = f(h(x) - 5) = \\ &= f(\sqrt{x} - 5) = (\sqrt{x} - 5)^3 + 8 \end{aligned}$$

Practice: Exercise 3

A spherical balloon is being inflated. The radius of the balloon is increasing at a rate of 3cm/s.

1. Find a function f that models the radius as a function of time t , in seconds.

$$f(t) = 3t$$

2. Find a function g that models the volume as a function of the radius r , in cm.

$$g(r) = \frac{4\pi r^3}{3} \quad (\text{formula of volume of a sphere})$$

3. Find and interpret $g \circ f$.

$$(g \circ f)(t) = g(f(t)) = \frac{4\pi (3t)^3}{3} = 36\pi t^3$$

finds $\downarrow$ the volume in term of time.

6. One-to-one Functions

A function f is **one-to-one** if there is a one-to-one correspondence between the inputs and the outputs. In other words, f never takes on the same values twice.

Examples: Is f one-to-one?

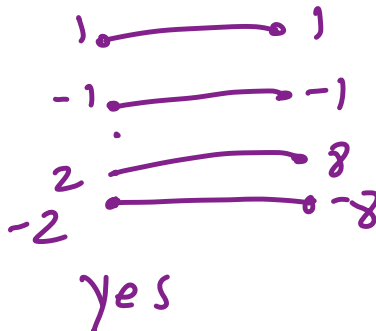
$$f(x) = x^2$$

$$f(1) = 1$$

$$f(-1) = 1$$



$$f(x) = x^3$$



$$f(1) = 1$$

$$f(-1) = -1$$

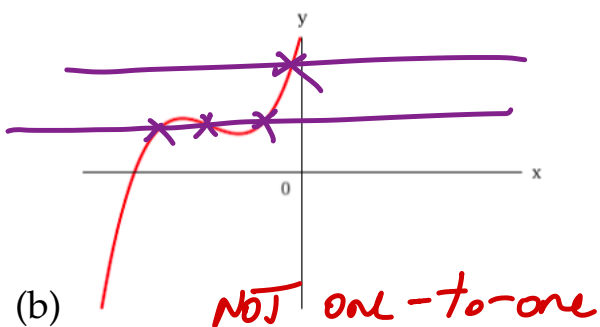
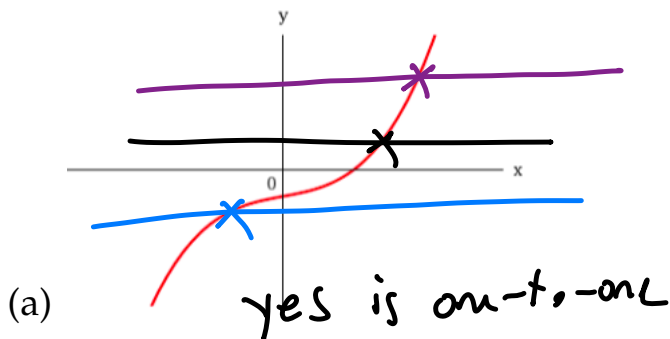
$$f(2) = 8$$

$$f(-2) = -8$$

How to determine if a function f is one-to-one - Graphically

Determine if the functions f and g are one-to-one.

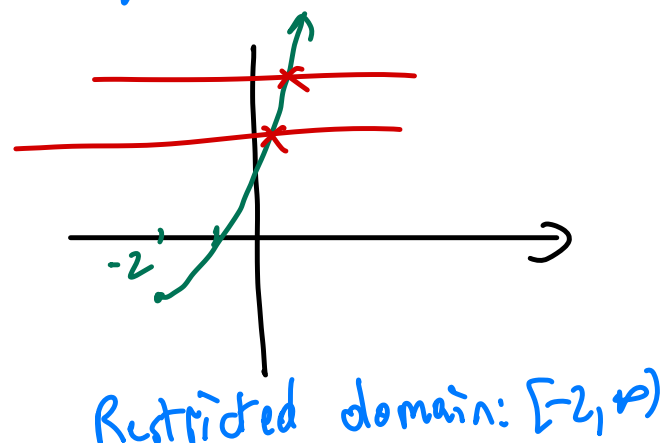
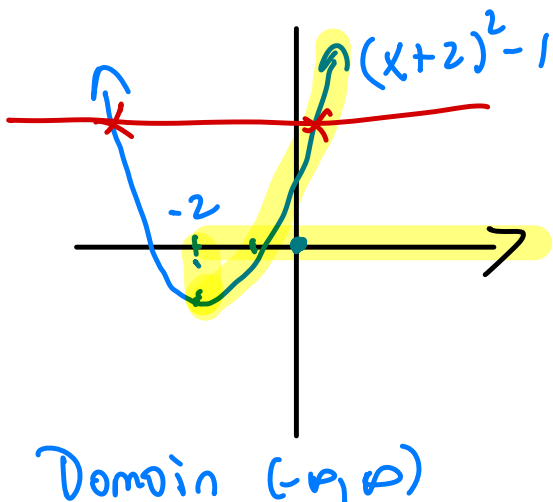
Horizontal line test



Practice - Restricted Domain

Graph the function $f(x) = (x + 2)^2 - 1$ and decide if it is one-to-one on its entire domain.

x^2 shifted left 2 units and down 1 unit



7. Inverse Functions

f^{-1} means the inverse of f

A **one-to-one function** with domain A and range B has an **inverse function** which takes on **as domain** the **range of f** and gives as output the **domain of f** . In other words the inverse of a function f undoes what f does. The inverse, denoted as **f^{-1}** takes as input the output of f and gives as output the input of f , i.e. if $f(x) = y$ then $f^{-1}(y) = x$.

For example if f is the function that takes the input x multiplies it by 5, adds 2 then takes the 5th power of the result, what would the inverse f^{-1} have to be to undo what f did?

$$f(x) = (5x + 2)^5$$

$$f^{-1}(x) = \frac{\sqrt[5]{x} - 2}{5}$$

the opposite in reverse order

The inverse function f^{-1} reverses the effect of f !

INVERSE FUNCTION PROPERTY

Let f be a one-to-one function with domain A and range B . The inverse function f^{-1} satisfies the following **cancellation properties**:

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

Conversely, any function f^{-1} satisfying these equations is the inverse of f .

Practice: Are f and g inverses of each other?

$$f(x) = 8 - 4x, \quad g(x) = \frac{8 - x}{4}$$

Check: $(f \circ g)(x) \stackrel{?}{=} x$
 $(g \circ f)(x) \stackrel{?}{=} x$

$$(f \circ g)(x) = f(g(x)) = 8 - 4g(x) = 8 - 4\left(\frac{8 - x}{4}\right) = \cancel{8} - \cancel{8} + x = x$$

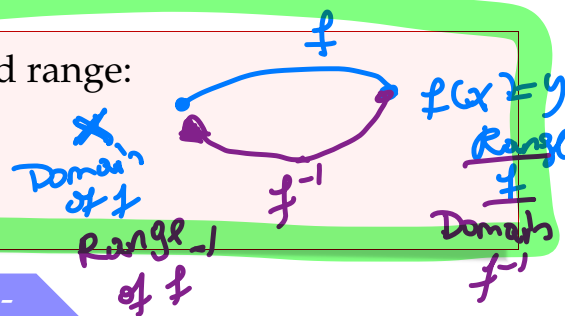
$$(g \circ f)(x) = g(f(x)) = \frac{8 - f(x)}{4} = \frac{8 - (8 - 4x)}{4} =$$

$$= \frac{\cancel{8} - \cancel{8} + 4x}{4} = \frac{4x}{4} = x$$

yes!

NOTE: The function and its inverse switch domain and range:

Domain of $f = \text{Range of } f^{-1}$
 Range of $f = \text{Domain of } f^{-1}$



8. How to find the inverse of a one-to-one function - Algebraically

Let f be a one-to-one function. Then $f(x) = y$ for any x in the domain of f and f has an inverse denoted f^{-1} .

Example: Let $f(x) = \frac{x-9}{x+9}$.

(a) Is f one-to-one?

yes (pass HTL)

(b) Find the domain of f .

$x \neq -9 \quad (-\infty, -9) \cup (-9, \infty)$

(c) Can you find the range of f without graphing it?

Not really

(d) Find the inverse of f .

$$f^{-1}(x) = \frac{-9-9x}{x-1}$$

Domain $f^{-1} : x \neq 1$
 $(-\infty, 1) \cup (1, \infty)$

(e) Can you find the range of f now using the information you have about f^{-1} ?

Range of $f = \text{domain of } f^{-1}$
 $(-\infty, 1) \cup (1, \infty)$

steps to find f^{-1}

1) Write $f(x) = \frac{x-9}{x+9}$

$$\text{as } y = \frac{x-9}{x+9}$$

2) Switch x and y

$$x = \frac{y-9}{y+9}$$

3) Solve for y

$$x(y+9) = y-9$$

$$xy + 9x = y - 9$$

$$xy - y = -9 - 9x$$

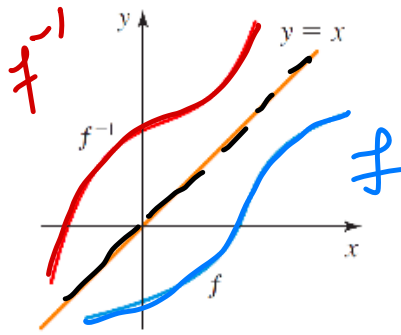
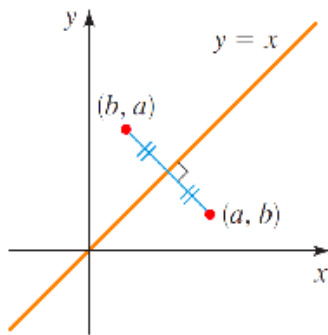
$$y(x-1) = -9 - 9x$$

$$y = \frac{-9-9x}{x-1} = f^{-1}(x)$$

9. How to find the inverse of a one-to-one function - Graphically

Let f be a one-to-one function. Then $f(x) = y$ for any x in the domain of f has an inverse f^{-1} . A point $(\underline{a}, \underline{b})$ on the graph of f is going to become $(\underline{b}, \underline{a})$ on the graph of f^{-1} since the input is switched with the output.

The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.



Example: Given the graph of f below find the graph of f^{-1} .

