Identity vs. Conditional Equation

A trigonometric identity is an equation involving trigonometric functions that is true for all values in the domain of the variable. Unlike a conditional equation, which is only true for specific values of the variable, an identity is true for all possible inputs where the expressions are defined.
For example, $\sin^2\theta + \cos^2\theta = 1$ is an identity because it is true for all values of θ. In contrast, $\sin\theta = \frac{1}{2}$ is a conditional equation because it is only true for specific values of θ, such as $\theta = \frac{\pi}{6} + 2\pi n$ or $\theta = \frac{5\pi}{6} + 2\pi n$, where n is an integer.

Purpose of Knowing Identities

- They help solve trigonometric equations by converting them into more manageable forms.
- They enable us to integrate or differentiate complicated functions involving trigonometric terms.

Guidelines for Verifying Identities

1. Work with one side of the equation at a time, typically the more complex side.
2. Never perform operations that assume the identity is true (like cross-multiplication).
3. Look for opportunities to use fundamental identities to simplify expressions.
4. Convert all functions to sines and cosines when dealing with multiple different functions.
5. Factor expressions when possible.

6. Look for common denominators when dealing with fractions.
7. Remember that verification is complete when one side is transformed to match the other.

Examples of Verifying Trigonometric Identities

Example 1: $\dfrac{\sin^2\theta + \cos^2\theta}{\cos^2\theta \cdot \sec^2\theta} = 1$

Solution:
Start with the left side and simplify:

$$\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta \cdot \sec^2\theta}$$

Using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$:

$$\frac{1}{\cos^2\theta \cdot \sec^2\theta}$$

Using the reciprocal identity $\sec\theta = \frac{1}{\cos\theta}$, we have $\sec^2\theta = \frac{1}{\cos^2\theta}$:

$$\frac{1}{\cos^2\theta \cdot \frac{1}{\cos^2\theta}} = \frac{1}{1} = 1$$

Therefore, the identity is verified.

Example 2: $2\csc^2\beta = \dfrac{1}{1-\cos\beta} + \dfrac{1}{1+\cos\beta}$

Solution:
Start with the right side:

$$\frac{1}{1-\cos\beta} + \frac{1}{1+\cos\beta}$$

Find a common denominator:

$$\frac{1+\cos\beta}{(1-\cos\beta)(1+\cos\beta)} + \frac{1-\cos\beta}{(1+\cos\beta)(1-\cos\beta)}$$

Simplify the denominators using difference of squares:
$(1-\cos\beta)(1+\cos\beta) = 1^2 - (\cos\beta)^2 = 1-\cos^2\beta = \sin^2\beta$:

$$\frac{1+\cos\beta}{\sin^2\beta} + \frac{1-\cos\beta}{\sin^2\beta} = \frac{1+\cos\beta+1-\cos\beta}{\sin^2\beta} = \frac{2}{\sin^2\beta}$$

Using the reciprocal identity $\csc\beta = \frac{1}{\sin\beta}$, we have $\csc^2\beta = \frac{1}{\sin^2\beta}$:

$$\frac{2}{\sin^2\beta} = 2\cdot\frac{1}{\sin^2\beta} = 2\csc^2\beta$$

Therefore, the identity is verified.

Example 3: $(sec^2x-1)(sin^2x-1) = -\sin^2 x$

Solution:
Start with the left side:

$$(sec^2x-1)(sin^2x-1)$$

Using the Pythagorean identity $\sec^2 x - 1 = \tan^2 x$:

$$(\tan^2 x)(sin^2x-1)$$

Using the Pythagorean identity $\sin^2 x - 1 = -\cos^2 x$:

$$(\tan^2 x)(-\cos^2 x) = -\tan^2 x\cos^2 x$$

Using the identity $\tan x = \frac{\sin x}{\cos x}$, we have $\tan^2 x = \frac{\sin^2 x}{\cos^2 x}$:

$$-\frac{\sin^2 x}{\cos^2 x}\cdot\cos^2 x = -\sin^2 x$$

Therefore, the identity is verified.

Example 4: $\tan x + \cot x = \sec x\csc x$

Solution:
We'll convert everything to sine and cosine.
Start with the left side:

$$\tan x + \cot x$$

Using the quotient identities $\tan x = \frac{\sin x}{\cos x}$ and $\cot x = \frac{\cos x}{\sin x}$:

$$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$

Finding a common denominator:

$$\frac{\sin^2 x + \cos^2 x}{\cos x \sin x}$$

Using the Pythagorean identity $\sin^2 x + \cos^2 x = 1$:

$$\frac{1}{\cos x \sin x}$$

For the right side, using the reciprocal identities $\sec x = \frac{1}{\cos x}$ and $\csc x = \frac{1}{\sin x}$:

$$\sec x \csc x = \frac{1}{\cos x} \cdot \frac{1}{\sin x} = \frac{1}{\cos x \sin x}$$

Since both sides are now identical, the identity is verified.

Example 5: $\sec x + \tan x = \frac{\cos x}{1 - \sin x}$

Solution:

Start with the left side:

$$\sec x + \tan x$$

Using the reciprocal identity $\sec x = \frac{1}{\cos x}$ and the quotient identity $\tan x = \frac{\sin x}{\cos x}$:

$$\frac{1}{\cos x} + \frac{\sin x}{\cos x} = \frac{1 + \sin x}{\cos x}$$

Now we need to show that $\frac{1+\sin x}{\cos x} = \frac{\cos x}{1-\sin x}$

Let's multiply both the numerator and denominator of $\frac{\cos x}{1-\sin x}$ by $(1 + \sin x)$:

$$\frac{\cos x}{1 - \sin x} = \frac{\cos x(1 + \sin x)}{(1 - \sin x)(1 + \sin x)}$$

Using the difference of squares formula, $(1 - \sin x)(1 + \sin x) = 1^2 - (\sin x)^2 = 1 - \sin^2 x = \cos^2 x$:

$$\frac{\cos x(1 + \sin x)}{\cos^2 x} = \frac{1 + \sin x}{\cos x}$$

Therefore, the identity is verified.

Example 6: $\dfrac{\cot^2\theta}{1+\csc\theta} = \dfrac{1-\sin\theta}{\sin\theta}$

Solution:
We'll work with both sides separately and show they are equal.
Starting with the left side:

$$\frac{\cot^2\theta}{1+\csc\theta}$$

Using the quotient identity $\cot\theta = \frac{\cos\theta}{\sin\theta}$ and the reciprocal identity $\csc\theta = \frac{1}{\sin\theta}$:

$$\frac{\left(\frac{\cos\theta}{\sin\theta}\right)^2}{1+\frac{1}{\sin\theta}} = \frac{\frac{\cos^2\theta}{\sin^2\theta}}{\frac{\sin\theta+1}{\sin\theta}} = \frac{\cos^2\theta}{\sin^2\theta}\cdot\frac{\sin\theta}{\sin\theta+1} = \frac{\cos^2\theta}{\sin\theta(\sin\theta+1)}$$

Now, working with the right side:

$$\frac{1-\sin\theta}{\sin\theta}$$

Using the Pythagorean identity $\cos^2\theta = 1-\sin^2\theta$, we can rewrite $1-\sin\theta$ as follows:

$$1-\sin\theta = \frac{1-\sin\theta}{1} = \frac{1-\sin\theta}{1}\cdot\frac{1+\sin\theta}{1+\sin\theta} = \frac{(1-\sin\theta)(1+\sin\theta)}{1+\sin\theta} = \frac{1-\sin^2\theta}{1+\sin\theta} = \frac{\cos}{1+\mathrm{s}}$$

Substituting this back into the right side:

$$\frac{1-\sin\theta}{\sin\theta} = \frac{\frac{\cos^2\theta}{1+\sin\theta}}{\sin\theta} = \frac{\cos^2\theta}{\sin\theta(1+\sin\theta)}$$

Since both sides are now identical, the identity is verified.