Mth 146: Strategy for Testing Series (11.7)

Reference Sheet

Big idea: Don't run through the tests in order until one works. Instead, **classify the series by its form**, just like choosing an integration technique. **Always glance at** $\lim a_n$ **first** — if it is clearly not 0, you're done.

1. Test for Divergence

Use when: you can see that $\lim_{n\to\infty} a_n$ may be different from 0.
Test: If $\lim_{n\to\infty} a_n \neq 0$ (or the limit does not exist), then $\sum a_n$ **diverges**.
Careful: If $\lim a_n = 0$, this test tells you *nothing* — try another test.

Example. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$. $\displaystyle\lim_{n\to\infty} \frac{n-1}{2n+1} = \lim_{n\to\infty} \frac{1-\frac{1}{n}}{2+\frac{1}{n}} = \frac{1}{2} \neq 0$, so the series **diverges**.

2. p-Series

Use when: the series has the form $\sum 1/n^p$.
Test: $\sum_{n=1}^{\infty} 1/n^p$ **converges** if $p > 1$ and **diverges** if $p \leq 1$.

Example. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ **converges** ($p = \frac{3}{2} > 1$), but $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ **diverges** ($p = \frac{1}{2} \leq 1$).

3. Geometric Series

Use when: the series has the form $\sum ar^{n-1}$ or $\sum ar^n$ (you may need some algebra to see it).
Test: $\sum_{n=1}^{\infty} ar^{n-1}$ **converges** to $a/(1-r)$ if $|r| < 1$, and **diverges** if $|r| \geq 1$.

Example. $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+1}}{4^n} = \frac{9}{4} + \frac{27}{16} + \frac{81}{64} + \cdots$ $a = \frac{9}{4}$, $r = \frac{3}{4}$, $|r| < 1$, so it **converges** to $\frac{9/4}{1-3/4} = 9$.

4. Comparison Tests (Direct and Limit)

Use when: the series looks *similar* to a p-series or a geometric series.
• a_n rational or algebraic in n (roots of polynomials) $\Rightarrow$ compare with a **p-series**; choose b_n by keeping only the **highest powers of** n on top and bottom.
• a_n involves something like r^n $\Rightarrow$ compare with a **geometric series**.
Direct Comparison: If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges. If $a_n \geq b_n \geq 0$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.
Limit Comparison: If $\lim_{n\to\infty} a_n/b_n = c$ with $0 < c < \infty$, then $\sum a_n$ and $\sum b_n$ *both* converge or *both* diverge.
Note: These tests need **positive terms**. If some terms are negative, apply them to $\sum |a_n|$ to test for absolute convergence.

Example. (Limit Comparison) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$. Keep the highest powers: $b_n = \dfrac{\sqrt{n^3}}{3n^3} = \dfrac{1}{3n^{3/2}}$; use $b_n = \dfrac{1}{n^{3/2}}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{3/2}\sqrt{n^3+1}}{3n^3+4n^2+2} = \lim_{n\to\infty} \frac{\sqrt{1+1/n^3}}{3+4/n+2/n^3} = \frac{1}{3},$$

which is positive and finite. Since $\sum 1/n^{3/2}$ converges ($p = \frac{3}{2} > 1$), the given series **converges**.

Example. (Direct Comparison) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+3^n}$. $\dfrac{1}{2+3^n} < \dfrac{1}{3^n} = \left(\dfrac{1}{3}\right)^n$, and $\sum(1/3)^n$ is a convergent geometric series ($|r| = \frac{1}{3} < 1$), so the series **converges**.

5. Alternating Series Test

Use when: the series has the form $\sum(-1)^{n-1}b_n$ or $\sum(-1)^n b_n$ with $b_n > 0$.
Test: If (i) $b_{n+1} \le b_n$ for all n and (ii) $\lim_{n\to\infty} b_n = 0$, then the series **converges**.
Tip: If $\sum |a_n| = \sum b_n$ converges, the series is **absolutely convergent** (and therefore convergent). If $\sum b_n$ diverges but the AST works, it is **conditionally convergent**.

Example. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{n^4+1}$. $|a_n| = \dfrac{n^2}{n^4+1} < \dfrac{n^2}{n^4} = \dfrac{1}{n^2}$, and $\sum 1/n^2$ converges, so $\sum |a_n|$ converges. The series is **absolutely convergent**, hence convergent.

Example. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges by the AST ($\frac{1}{n+1} < \frac{1}{n}$ and $\frac{1}{n} \to 0$), but $\sum \frac{1}{n}$ diverges, so it is **conditionally convergent**.

6. Ratio Test

Use when: the terms involve **factorials** or other products, including a **constant raised to the nth power** (like 2^n, 5^n).
Test: Let $L = \lim_{n\to\infty} |a_{n+1}/a_n|$. $L < 1 \Rightarrow$ **absolutely convergent**; $L > 1$ or $L = \infty \Rightarrow$ **divergent**; $L = 1 \Rightarrow$ **inconclusive**.
Careful: $L = 1$ for *every* p-series, and for all rational or algebraic a_n — so don't use the Ratio Test on those.

Example. $\displaystyle\sum_{k=1}^{\infty}\frac{2^k}{k!}$. $\left|\dfrac{a_{k+1}}{a_k}\right| = \dfrac{2^{k+1}}{(k+1)!}\cdot\dfrac{k!}{2^k} = \dfrac{2}{k+1} \to 0 = L < 1$, so the series is **absolutely convergent**.

7. Root Test

Use when: a_n has the form $(b_n)^n$ — the whole term is raised to the nth power.
Test: Let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$. The conclusions are the same as the Ratio Test: $L < 1 \Rightarrow$ absolutely convergent; $L > 1$ or $\infty \Rightarrow$ divergent; $L = 1 \Rightarrow$ inconclusive.

Example. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$. $\sqrt[n]{|a_n|} = \dfrac{2n+3}{3n+2} = \dfrac{2+3/n}{3+2/n} \to \dfrac{2}{3} = L < 1$, so the series is **absolutely convergent**.

8. Integral Test

Use when: $a_n = f(n)$, where $\int_1^{\infty} f(x)\,dx$ is **easy to evaluate**.
Test: If f is **continuous, positive, and decreasing** on $[1,\infty)$, then $\sum a_n$ and $\int_1^{\infty} f(x)\,dx$ *both* converge or *both* diverge.

Example. $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$. $f(x) = xe^{-x^2}$ is continuous, positive, and decreasing for $x \ge 1$ (since $f'(x) = (1-2x^2)e^{-x^2} < 0$). With $u = x^2$, $\displaystyle\int_1^{\infty} xe^{-x^2}\,dx = \lim_{t\to\infty}\left[-\tfrac{1}{2}e^{-x^2}\right]_1^t = \frac{1}{2e}$ is finite, so the series **converges**.

Quick pick — what do you see in a_n?

$\lim a_n$ clearly $\neq 0$	$\Rightarrow$ Divergence Test	$(-1)^n$ or $(-1)^{n-1}$	$\Rightarrow$ check $\sum \lvert a_n\rvert$, then AST
$1/n^p$	$\Rightarrow$ p-series	Factorials, c^n	$\Rightarrow$ Ratio Test
ar^n	$\Rightarrow$ Geometric	Whole term to the nth power	$\Rightarrow$ Root Test
Rational / roots in n	$\Rightarrow$ Compare with p-series	$f(n)$ with an easy integral	$\Rightarrow$ Integral Test
Close to r^n	$\Rightarrow$ Compare with geometric		