Mth 146: Day 21

10.1 Curves Defined by Parametric Equations
10.2 Calculus with Parametric Curves

1 Part 1: New Concepts & Worked Examples

10.1 Curves Defined by Parametric Equations

Some curves fail the Vertical Line Test, so we can't write them as $y = f(x)$. Instead, we describe both x and y in terms of a third variable.

Definition. If x and y are both functions of a third variable t, called the _____,

$$x = f(t) \qquad y = g(t)$$

these are called _____. As t varies, the point $(x, y) = (f(t), g(t))$ traces out a _____.

Think of t as **time**: $(f(t), g(t))$ is the position of a moving particle at time t.

Example 1. Sketch and identify the curve
$x = t^2 - 2t, \quad y = t + 1.$

Eliminating the parameter (finding an equation in only x and y) helps identify the shape, but we _____.

If we restrict $a \leq t \leq b$, the curve has **initial point** _____ and **terminal point** _____.

Example: In Example 1 with $0 \leq t \leq 4$, the curve starts at _____ and ends at _____.

Example 2. What curve is given by $x = \sin 2t,\ y = \cos 2t,\ 0 \leq t \leq 2\pi$?

Note: Different parametric equations can trace the _____ in different ways (direction, speed, number of times around).

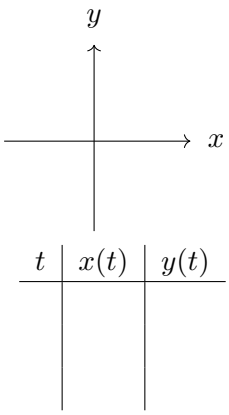
Example 3. Consider the three parametrizations

$$\begin{aligned}
&\text{(a)} \quad x = \sin t, \quad y = \cos t, \quad 0 \le t \le \pi \\
&\text{(b)} \quad x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi \\
&\text{(c)} \quad x = \sin 2t, \quad y = \cos 2t, \quad 0 \le t \le 2\pi
\end{aligned}$$

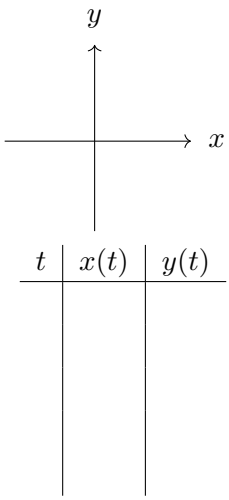
All three trace portions of the same curve.

- What is the curve?
- What is the initial position and the direction traced by each one?
- How many times is it traced?

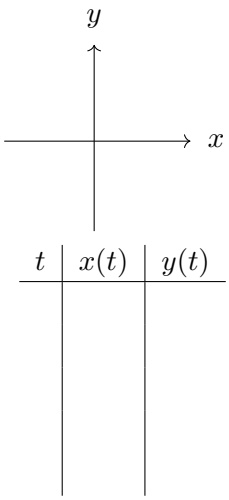
(a)



(b)



(c)



Curve: _____

(a) Start: _____ Direction: _____ Times: _____

(b) Start: _____ Direction: _____ Times: _____

(c) Start: _____ Direction: _____ Times: _____

Circle with center (h, k) and radius r:

$$x = _____ \qquad y = _____ \qquad 0 \le t \le 2\pi$$

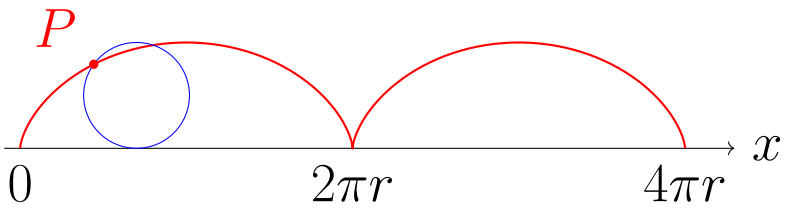
(traced once, counterclockwise)

Example 4. Sketch the curve $x = \sin t,\ y = \sin^2 t$.

Graphing tip: Any curve $x = g(y)$ can be graphed on a calculator using $x = g(t),\ y = t$. Likewise, $y = f(x)$ is the parametric curve $x = t,\ y = f(t)$.

The Cycloid

A _____ is the curve traced by a point P on the rim of a circle of radius r as the circle rolls along the x-axis (like a pebble stuck in a tire).



Using the angle of rotation θ as the parameter, the cycloid is

$$x = _____ \qquad y = _____$$

One arch comes from one full turn of the circle: _____.

10.2 Calculus with Parametric Curves

Tangents

Slope of the tangent line. If $x = f(t)$, $y = g(t)$, then

$$\frac{dy}{dx} = _____ \qquad \text{if } \frac{dx}{dt} \neq 0.$$

- **Horizontal tangent** when _____ (and $dx/dt \neq 0$).
- **Vertical tangent** when _____ (and $dy/dt \neq 0$).

Second derivative. Replace y by $\dfrac{dy}{dx}$:

$$\frac{d^2y}{dx^2} = _____$$

Careful: $\dfrac{d^2y}{dx^2} \neq \dfrac{d^2y/dt^2}{d^2x/dt^2}$.

Example 5. A curve C is defined by $x = t^2$, $y = t^3 - 3t$.
(a) Show that C has two tangents at $(3, 0)$ and find their equations.
(b) Find the points where the tangent is horizontal or vertical.
(c) Determine where the curve is concave up or down.
(d) Sketch the curve.

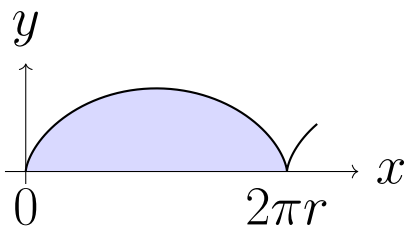
Example 6. (a) Find the tangent to the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$ at $\theta = \pi/3$.
(b) Where is the tangent horizontal?

If the curve $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$, is traced out **once**, the area under it is found with the Substitution Rule:

$$A = \int_a^b y\,dx = _____ \qquad [\text{or } _____]$$

The t-limits match the x-limits: when $x = a$, t is either α or β; when $x = b$, t is the other value.

Example 7. Find the area under one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.



Theorem. If C is given by $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, with f', g' continuous, and C is traversed _____ as t goes from α to β, then

$$L = _____$$

Example 8. Find the length of the unit circle using (a) $x = \cos t,\ y = \sin t,\ 0 \leq t \leq 2\pi$ and (b) $x = \sin 2t,\ y = \cos 2t,\ 0 \leq t \leq 2\pi$.

Speed

If $(x(t), y(t))$ is the position of a particle at time t, its **speed** is

$$v(t) = _____$$

Example 9. A particle moves by $x = 2t + 3$, $y = 4t^2$, $t \geq 0$. Find its speed at the point $(5, 4)$.

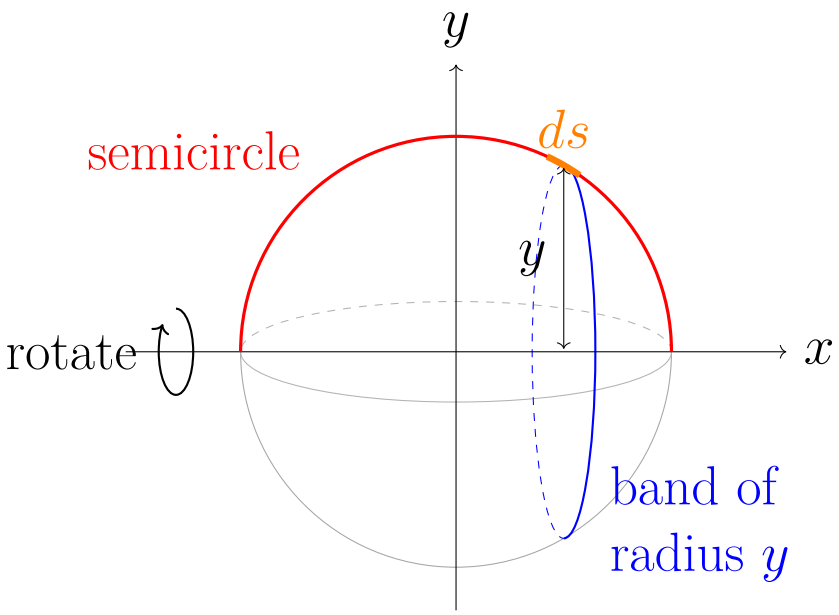
Surface Area

If C: $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, with $g(t) \geq 0$, is traversed once and rotated about the x**-axis**, the surface area is

$$S = _____$$

Why $2\pi y$? Each tiny piece of the curve (length ds) sweeps out a thin band. The band's radius is y, so its area is about $2\pi y\, ds$.

Example 10. Show that the surface area of a sphere of radius r is $4\pi r^2$.



2 Part 2: Now You Try It

Final answers are given as footnotes at the bottom of the page — only the final answer, not the steps, so use them to check your work rather than replace it.

1. Eliminate the parameter to find a Cartesian equation: $x = 3 - 4t,\ y = 2 - 3t$.
   [1]

2. Eliminate the parameter and describe the motion: $x = 2\cos t,\ y = 3\sin t,\ 0 \le t \le 2\pi$.
   [2]

[1] Answer: $y = \frac{3}{4}x - \frac{1}{4}$ (a line).

[2] Answer: $\frac{x^2}{4} + \frac{y^2}{9} = 1$; ellipse traced once counterclockwise from $(2, 0)$.

3. Eliminate the parameter: $x = \sqrt{t},\ y = 1 - t$. Which part of the curve is traced?

3

4. Find an equation of the tangent line to $x = t^4 + 1,\ y = t^3 + t$ at $t = -1$.

4

5. Find the points where $x = t^3 - 3t,\ y = t^2 - 3$ has a horizontal or vertical tangent.

5

[3] Answer: $y = 1 - x^2$, only the part with $x \geq 0$.
[4] Answer: $y = -x$.
[5] Answer: horizontal at $(0, -3)$; vertical at $(-2, -2)$ and $(2, -2)$.

6. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for $x = t^2 + 1,\ y = t^2 + t$. For which t is the curve concave up?

6

7. Find the length of the curve $x = 1 + 3t^2,\ y = 4 + 2t^3,\ 0 \le t \le 1$.

7

8. Find the speed of a particle with position $x = t^2,\ y = t^3$ at $t = 1$.

8

[6] Answer: $\dfrac{dy}{dx} = 1 + \dfrac{1}{2t}$, $\dfrac{d^2y}{dx^2} = -\dfrac{1}{4t^3}$; concave up for $t < 0$.
[7] Answer: $L = 4\sqrt{2} - 2 \approx 3.66$.
[8] Answer: $\sqrt{13} \approx 3.61$.

9. Find the length of one arch of the cycloid $x = \theta - \sin\theta,\ y = 1 - \cos\theta$ (so $r = 1$). *Hint:* $1 - \cos\theta = 2\sin^2(\theta/2)$.

9

[9] Answer: $L = 8$ (in general, $8r$).

3 Part 3: Reflection

For each item below, rate your comfort level and jot a quick note: What clicked? What's still fuzzy? What would help?

1. **Sketch a parametric curve by plotting points and showing the direction of motion.** 1 2 3 4 5

2. **Eliminate the parameter and identify which part of the curve is traced.** 1 2 3 4 5

3. **Find dy/dx and d^2y/dx^2, and horizontal/vertical tangents, for a parametric curve.** 1 2 3 4 5

4. **Set up and evaluate the arc length of a parametric curve (traced once).** 1 2 3 4 5

5. **Find the speed of a moving particle and the surface area of a rotated parametric curve.** 1 2 3 4 5