Example 3. $\int e^x \sin(x)\,dx$ =

$$\int e^x \sin x\,dx = -e^x\cos x + \int e^x \cos x\,dx$$

$$= -e^x\cos x + e^x \sin x - \int e^x \sin x\,dx$$

add $\int e^x \sin x\,dx$ to both sides

$$2\int e^x \sin x\,dx = -e^x\cos x + e^x \sin x$$

$$\int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C$$

Cyclical IBP

IBP

LIPET

$\int u\,dv = uv - \int v\,du$

$u = e^x$ $\quad dv = \sin x$

$du = e^x dx$ $\quad v = -\cos x$

$\int e^x \cos x\,dx$ IBP

$u = e^x$ $\quad dv = \cos x$

$du = e^x dx$ $\quad v = \sin x$

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

How to choose u and dv wisely

L (logarithm)

I (inverses: $\arctan x$, $\arccos$

P (polynomials: x, x^2, θ, θ^3, t, ...)

E (exponentials: e^x, 5^x, e^{2x}, ...)

T (trig ($\sin x$, $\sec x$, $\tan x$)

Choose u in this order!!!

Example 4. $\int \ln(x)\,dx$

IBP

$\int u\,dv = uv - \int v\,du$

LIPET

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

$\int \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}\,dx$

$= x\ln x - x + C$

Example 5. $\int t(5)^t\,dt$

IBP

$\int u\,dv = uv - \int v\,du$

~~LIPET~~

$u = t \qquad dv = 5^t\,dt$

$du = dt \qquad v = \frac{5^t}{\ln 5}$

$= \frac{t\,5^t}{\ln 5} - \int \frac{5^t}{\ln 5}\,dt$

$= \frac{t\,5^t}{\ln 5} - \frac{1}{\ln 5}\int 5^t\,dt$

$= \frac{t5^t}{\ln 5} - \frac{1}{\ln 5}\,\frac{5^t}{\ln 5} + C$

$= \frac{5^t}{\ln 5}\left(t - \frac{1}{\ln 5}\right) + C$

Example 6. $\int_0^1 \arctan(t)\,dt$

$$\int \arctan(t)\,dt$$

$$= t\arctan t - \int t\cdot\frac{1}{1+t^2}\,dt$$

$$= t\arctan(t) - \underbrace{\int \frac{t}{1+t^2}\,dt}_{u\text{-sub}}$$

$$= t\arctan(t) - \frac{1}{2}\ln(1+t^2) + C$$

$$\int_0^1 \arctan(t) =$$

$$= t\arctan(t) - \frac{1}{2}\ln(1+t^2)\Big|_0^1$$

$$= 1\cdot\arctan(1) - \frac{1}{2}\ln(1+1^2) - \left(0\cdot\arctan 0 - \frac{1}{2}\ln(1+0^2)\right)$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln(2)$$

IBP

$\int u\,dv = uv - \int v\,du$

LIPET

$u = \arctan t \qquad dv = dt$

$du = \frac{1}{1+t^2}dt \qquad v = t$

$$\int \frac{t}{1+t^2}\,dt = \qquad u = 1+t^2$$

$$= \frac{1}{2}\int \frac{1}{u}\,du \qquad du = 2t\,dt$$

$$= \frac{1}{2}\ln|u| + C \qquad \frac{1}{2}du = t\,dt$$

$$= \frac{1}{2}\ln(1+t^2) + C$$

Example 7. $\int x\ln(1+x)\,dx$

$$= \frac{x^2}{2}\ln(1+x) - \int \frac{x^2}{2}\cdot\frac{1}{1+x}\,dx$$

$$= \frac{x^2}{2}\ln(1+x) - \frac{1}{2}\int \frac{x^2}{1+x}\,dx$$

$$= \frac{x^2}{2}\ln(1+x) - \frac{1}{2}\left(\frac{(1+x)^2}{2} - 2(1+x) + \ln|1+x|\right) + C$$

IBP

$\int u\,dv = uv - \int v\,du$

LIPET

$u = \ln(1+x) \qquad dv = x\,dx$

$du = \frac{1}{1+x}dx \qquad v = \frac{x^2}{2}$

$$\int \frac{x^2}{1+x}\,dx$$

$u = 1+x$, $du = dx$, $x = u-1$, $x^2 = (u-1)^2$

$$\int \frac{(u-1)^2}{u}\,du$$

$$= \int \frac{u^2-2u+1}{u}\,du$$

$$= \int \frac{u^2}{u}\,du - 2\int \frac{u}{u}\,du + \int \frac{1}{u}\,du$$

$$= \int u\,du - 2\int 1\,du + \ln|u|$$

$$= \frac{u^2}{2} - 2u + \ln|u| + C$$

2 Part 2: Your Turn — Practice

Try the following individually or in small groups. Final answers are given as footnotes at the bottom of the page — only the final answer, not the steps, so use them to check your work rather than replace it.

IBP: $u = x^2$ $dv =$

1. $\displaystyle\int_0^{\pi/2} x^2 \cos(x)\, dx$.[1]

2. $\displaystyle\int e^x \arctan(e^x)\, dx$.[2]

u-sub

$u = e^x$

$du = e^x dx$

$= \int \arctan u \, du$

3. $\displaystyle\int x^2 \tan(x^3)\, dx$.[3]

u sub

$u = x^3$

$du = 3x^2 dx$

$\frac{du}{3} = x^2 dx$

$\int \tan u \, du$

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[1] Answer: $\frac{\pi^2}{4} - 2$.

[2] Answer: $e^x \arctan(e^x) - \frac{1}{2}\ln(1 + e^{2x}) + C$.

[3] Answer: $-\frac{1}{3}\ln|\cos(x^3)| + C$.

4. $\int_0^2 e^{-2e^x+2x}\,dx$.[4]

$$\int e^{-2e^x} \cdot e^{2x}\,dx$$
$$= \int e^{-2e^x} \cdot e^x \cdot e^x\,dx$$
$u = e^x$

Start with u-sub: $u = \ln x$, $du = \frac{1}{x}\,dx$

5. $\int \frac{\sin(\ln(x))}{x^2}\,dx$.[5]

$$= \int \frac{\sin(\ln x)}{x} \cdot \frac{1}{x}\,dx = \int \frac{\sin(u)}{e^u}\,du$$

$u = \ln x$

$x = ?$

$e^u = e^{\ln x}$

$e^u = x$

$$= \int e^{-u} \sin u\,du$$

[4] Answer: $\frac{3}{4e^2} - e^{-2e^2}\left(\frac{e^2}{2} + \frac{1}{4}\right)$.

[5] Answer: $-\frac{\sin(\ln x) + \cos(\ln x)}{2x} + C$.

Mth 146: Day 3
Section 7.2 Trig Integrals

1 Part 1: New Concepts & Worked Examples

We will continue to learn techniques to find integrals of trigonometric functions. The goal is to use trigonometric identities and rewrite the integral in such a way that we can use a u-substitution, where the u-sub is a trigonometric function.

Often, simplifying/rewriting using trig identities will help. Some useful identities:

1. Pythagorean: $\sin^2 x + \cos^2 x = 1$
   $\tan^2 x + 1 = \sec^2 x$
   $\sin^2 x = 1 - \cos^2 x$
   $\cos^2 x = 1 - \sin^2 x$
   $\tan^2 x = \sec^2 x - 1$
2. Power Reducing: $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$
   $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$
3. Product to sum:
   $\sin(A)\cos(B) = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
   $\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
   $\cos(A)\cos(B) = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Sine $\times$ cosine integrals:

$\sin^3 x = \sin^2 x \sin x$

- Odd power of sine or cosine $\implies$ save one factor, then convert the rest using the Pythagorean identities
  $u = \cos x$ odd power of sine
- Only even powers $\implies$ use power reducing formulas to reduce powers until the integral can be computed

Tangent $\times$ secant integrals:

- Even secant $\implies$ $u = \tan(x)$, so save a $du = \sec^2(x)$ factor and use $\sec^2(x) = 1 + \tan^2(x)$
- Odd tangent $\implies$ $u = \sec(x)$, so save a $du = \sec(x)\tan(x)$ and use $\tan^2(x) = \sec^2(x) - 1$

Integrals involving powers of $\sin(x)$ and $\cos(x)$

even power sine
use power reducing
$\sin^2 x = \frac{1}{2}(1-\cos(2x))$

Example 1. $\int \sin^2(x)\,dx$ =

$$= \int \frac{1}{2}(1-\cos(2x))\,dx$$

$$= \frac{1}{2}\int 1\,dx - \frac{1}{2}\int \cos(2x)\,dx$$

$$= \frac{1}{2}x - \frac{1}{2}\,\frac{\sin(2x)}{2} + C$$

Example 2. $\int \sin^3(x)\,dx$ $= \int \sin^2 x \sin x\,dx$

$u = \cos x$
$du = -\sin x\,dx$

$\sin^2 x = 1-\cos^2 x$
$= 1-u^2$

$$= -\int 1-u^2\,du$$

$$= -\int 1\,du + \int u^2\,du$$

$$= -u + \frac{u^3}{3} + C$$

$$= -\cos x + \frac{\cos^3 x}{3} + C$$