$f(x) = (x+1)(x-2)(x+3)$

$x=1$ $x=2$ $x=-3$

2.3 Real Zeroes of Polynomial Functions

Next, we are interested in how to get the factored form of higher degree polynomials so that we can get the real zeros fast. We will review, factoring and long division to divide polynomials. Then we will use that to factor higher degree polynomial and interpret the remainder (if any).

Example 1: Divide the polynomial $f(x) = x^3 - 3x^2 + 4$ by $x - 2$ and use the result to factor the polynomial completely.

$$
\begin{array}{r|l}
 & x^2 - x - 2 \quad \leftarrow \text{quotient} \\
\hline
x-2 & x^3 - 3x^2 + 0x + 4 \rightarrow \text{dividend} \\
\text{divisor} & x^3 - 2x^2 \\
\hline
 & 0 \quad -x^2 + 0x + 4 \\
 & -x^2 + 2x \\
\hline
 & 0 \quad -2x + 4 \\
 & -2x + 4 \\
\hline
 & 0 \rightarrow \text{remainder}
\end{array}
$$

$\frac{x^3}{x} = x^2$

$x^2(x-2) = x^3 - 2x^2$

$-3x^2 - (-2x^2)$

$-3x^2 + 2x^2$

$-x^2$

$\frac{-x^2}{x} = -x$

$-x(x-2) = -x^2 + 2x$

$\frac{-2x}{x} = -2$

$-2(x-2) = -2x + 4$

$f(x) = x^3 - 3x^2 + 4 = (x-2)(x^2 - x - 2)$

$= (x-2)(x-2)(x+1)$

$f(x) = (x-2)^2(x+1)$

$x = 2$ zero $m = 2$ touches

$x = -1$ zero $m = 1$ crosses

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

divisor

remainder

$$\boxed{\underbrace{f(x)}_{\text{dividend}} = d(x)\underbrace{q(x)}_{\text{quotient}} + r(x)} \leftarrow \div d(x)$$

where $r(x) = 0$ *or* the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, then $d(x)$ *divides evenly* into $f(x)$.

The division algorithm can also be written as

$$\boxed{\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}}$$

Example 2: Divide $f(x) = (x^4 - 5x^2 + 4)$ by $(x^2 - 1)$.

$$
\begin{array}{r|l}
 & x^2 - 4 \\
\hline
x^2 - 1 & x^4 + 0x^3 - 5x^2 + 0x + 4 \\
 & x^4 \qquad\quad - x^2 \\
\hline
 & \qquad -4x^2 + 0x + 4 \\
 & \qquad -4x^2 \quad + 4 \\
\hline
 & \qquad\qquad 0
\end{array}
$$

$$f(x) = (x^2 - 1)(x^2 - 4) = (x-1)(x+1)(x-2)(x+2)$$

Example 3: Divide $f(x) = -2 + 3x - 5x^2 + 4x^3 + 2x^4$ by $x^2 + 2x - 3$

$$
\begin{array}{r|l}
 & 2x^2+1 \\
\hline
x^2+2x-3 & 2x^4+4x^3-5x^2+3x-2 \\
 & 2x^4+4x^3-6x^2 \\
\hline
 & x^2+3x-2 \\
 & x^2+2x-3 \\
\hline
 & x+1
\end{array}
$$

r

a) $f(x) = d(x)\,g(x) + r(x)$

or

b) $\dfrac{f(x)}{d(x)} = g(x) + \dfrac{r(x)}{d(x)}$

$r \neq 0$

a) $2x^4 + 4x^3 - 5x^2 + 3x - 2 = (x^2+2x-3)(2x^2+1) + x+1$

b) $\dfrac{2x^4+4x^3-5x^2+3x-2}{x^2+2x-3} = 2x^2+1 + \dfrac{x+1}{x^2+2x-3}$

* useful in Calc 2

Practice:

1. Divide the polynomial $f(x) = (x^3 - 9x^2 + 26x - 24)$ by $d(x) = (x - 2)$.

   (a) Does $x - 2$ divide evenly in $f(x)$? Yes

   (b) Calculate the value of $f(2)$.

$$
\begin{array}{r|l}
 & x^2 - 7x + 12 \\
\hline
x-2 & x^3 - 9x^2 + 26x - 24 \\
 & x^3 - 2x^2 \\
\hline
 & -7x^2 + 26x - 24 \\
 & -7x^2 + 14x \\
\hline
 & 12x - 24 \\
 & 12x - 24 \\
\hline
 & 0 \quad r=0
\end{array}
$$

$$f(2) = 2^3 - 9\cdot 2^2 + 26\cdot 2 - 24 = 8 - 36 + 52 - 24 = 0$$

2. Divide $f(x) = (x^4 - 5x^3 + 2x^2 + 8x - 6)$ by $d(x) = (x - 3)$

   (a) Does $x - 3$ divide evenly in $f(x)$?

   (b) Calculate the value of $f(3)$.

$$f(3) = -18$$

$$
\begin{array}{r|l}
 & x^3 - 2x^2 - 4x - 4 \\
\hline
x-3 & x^4 - 5x^3 + 2x^2 + 8x - 6 \\
 & x^4 - 3x^3 \\
\hline
 & -2x^3 + 2x^2 + 8x - 6 \\
 & -2x^3 + 6x^2 \\
\hline
 & -4x^2 + 8x - 6 \\
 & -4x^2 + 12x \\
\hline
 & -4x - 6 \\
 & -4x + 12 \\
\hline
 & -18 \leftarrow r \neq 0
\end{array}
$$

The Remainder and Factor Theorem

Remainder Theorem:
If a polynomial $f(x)$ is divided by $x - a$, then the remainder is equal to $f(a)$.

Factor Theorem:
$r=0$
A polynomial $f(x)$ has a factor $(x - a)$ if and only if $\boxed{f(a) = 0.}$

The Rational Zero Test

What if we are not given any roots or zeros of the polynomial but we are still asked to factor it. For example, in Example 1 we were told to divide $f(x) = x^3 - 3x^2 + 4$ by $x - 2$ and in doing that we were able to completely factor out $f(x)$. Next, we are interested in how to find potential zeros. This is the result of The Rational zero Test.

The Rational Zero Test

$\boxed{f(x) = (x-2)(x-2)(x+1)}$

leading coefficient

constant term

If the polynomial

$$* \quad \boxed{f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0}$$

has integer coefficients, then every rational zero of f has the form

$$\text{Rational zero} = \frac{p}{q}$$

reduced form

where p and q have no common factors other than 1, p is a factor of the constant term a_0, and
q is a factor of the leading coefficient a_n.

$$\boxed{\text{Possible rational zeros} = \frac{\text{factors of constant term} \to a_0}{\text{factors of leading coefficient} \to a_n}}$$

Example 4: Find the rational zeros of

$4 = 1 \cdot 4$
$4 = (-1)(-4)$
$4 = 2 \cdot 2$
$4 = (-2)(-2)$

1. $f(x) = 1 \cdot x^3 - 3x^2 + 4 = (x+1)(x-2)^2$

constant term $= 4$

factors of 4: $\pm 1, \pm 2, \pm 4$

leading coefficient $= 1$

factors of 1: ± 1

By Rational zero Test possible rational zeros are:

$\pm 1, \pm 2, \pm 4$

$f(1) \neq 0$ $\quad f(-1) = 0$ $\quad f(2) = 0$ $\quad f(-2) \neq 0$ $\quad f(4) \neq 0$ $\quad f(-4) \neq 0$

2. $h(x) = 2x^3 + 3x^2 - 8x + 3$

factors of constant term 3: $\pm 1, \pm 3$

factors of leading coefs 2: $\pm 1, \pm 2$

possible rational zeros:

$$\frac{\pm 1, \pm 3}{\pm 1, \pm 2}$$

$\pm 1, \pm 1/2, \pm 3, \pm 3/2$ possible zeros

$f(1) = 0$ $\quad f(1/2) = 0$ $\quad f(3) =$ $\quad f(3/2) =$

$f(-1) =$ $\quad f(-1/2) =$ $\quad f(-3) = 0$ $\quad f(-3/2) =$

The zeros are $1, 1/2, -3$

$h(x) = (x-1)(x+3)(x - 1/2) = (x-1)(x+3)(2x-1)$

3. $g(x) = 1 \cdot x^3 + x + 1$

lead term $\quad$ const term

possible rational zeros:

$$\frac{\pm 1}{\pm 1} = \pm 1$$

factors of 1 const term: ± 1

factors of 1 leading term: ± 1

$f(-1) = (-1)^3 + (-1) + 1 = -1$

$f(1) = 1 + 1 + 1 = 3$

None of them are zeros

it has an irrational root (or zero) between -1 and 0.

rational numbers: $1/2, -3/2, 3/1$ irrational numbers: $\pi, \sqrt{2}, e$

Real numbers: rational + irrational

A nth degree polynomial can have at most n real roots.
In the complex number system, every nth degree polynomial has exactly n zeros.

Recall Complex Numbers

$\sqrt{-1} = i$ or $i^2 = -1$

$\sqrt{-15} = \sqrt{(-1)(15)} = \boxed{\sqrt{-1}}\sqrt{15} = i\sqrt{15}$ → complex number

$\sqrt{-9} = \sqrt{(-1)9} = \sqrt{-1}\cdot\sqrt{9} = i3 = 3i$ → complex number

A complex number is a number of the form $a + bi$, where a and b are constants.

a is called the real part

b is called the imaginary part

EX $2 + 3i$ $a=2$, $b=3$ → complex number

EX $\sqrt{5}i = i\sqrt{5}$ $a=0$, $b=\sqrt{5}$ → complex number / pure imaginary

EX $-17 = -17 + 0i$, $a=-17$, $b=0$

→ real number

Note: -17 is a **both** a complex number **and** a real number.

Note Every real number is a complex number, but not every complex number is a real number.

Types of Zeros of a polynomial function f

Graphically real zeros are x-intercepts

- Real
  - rational: $f(x) = (2x+1)(x-1)$
    rational zeros: $-1/2$, 1
  - irrational: $f(x) = (x-1)(x^2-2)$
    $x=1$, $-\sqrt{2}, \sqrt{2}$ irrational zeros
    $x^2-2=0$
    $x^2=2$
    $x = \pm\sqrt{2}$
- Complex zeros : $2i$, $1-i$, $1+3i$

If $1-i$ is a zero then $1+i$ is zero also.

$1-i$ and $1+i$ are conjugates

Complex zeros come in pairs!

Fundamental Theorem of Algebra

Every polynomial function with degree $n \geq 1$ can be factored into n linear factors of the following form:

$f(x) = a(x - r_1) \cdot (x - r_2) \cdot (x - r_3) \cdot ... \cdot (x - r_n)$

where $r_1, r_2, r_3, ..., r_n$ are complex numbers and $a \neq 0$.

Example 5: Completely factor the polynomial over the complex numbers.

$f(x) = (x - 1)(x^2 + 4)(x^2 - 3)$

$x^2 - 3 = 0$

$x^2 = 3$

$x = \pm\sqrt{3}$

$(x - \sqrt{3})(x + \sqrt{3})$

$f(x) = (x-1)(x^2+4)(x-\sqrt{3})(x+\sqrt{3})$ (factored completely over the real numbers)

gives me complex roots

$x^2 + 4 = 0$

$x^2 = -4$

$x = \pm\sqrt{-4} = \pm i2$

$f(x) = (x-1)(x-\sqrt{3})(x+\sqrt{3})(x-2i)(x+2i)$

factored completely over the complex numbers

Example 6: List all possible rational zeros. Find all zeros and completely factor f over the complex numbers.

real: rational + irrat.

complex: come in pairs

$f(x) = 2x^4 + x^3 + 17x^2 + 9x - 9$

const. term: -9 factors: $\pm 1, \pm 3, \pm 9$

leading. term: 2 factors: $\pm 1, \pm 2$

possible rational zeros: $\pm 1, \pm 1/2, \pm 3, \pm 3/2, \pm 9, \pm 9/2$

Actual rational zeros: $x = -1$ and $x = 1/2$ (from the graph)

$x=-1 \rightarrow (x+1)$

$x=1/2 \rightarrow (x - 1/2) = (2x-1)$

$2x - 1 = 0$

$2x = 1$

$x = 1/2$

Multiply $(x+1)(2x-1) = 2x^2 - x + 2x - 1 = 2x^2 + x - 1$

$$
\begin{array}{r|l}
 & \quad x^2+9 \\
\hline
2x^2+x-1 & 2x^4+x^3+17x^2+9x-9 \\
 & 2x^4+x^3-x^2 \\
\hline
 & \qquad 18x^2+9x-9 \\
 & \qquad 18x^2+9x-9 \\
\hline
 & \qquad\qquad 0
\end{array}
$$

$$2x^4+x^3+17x^2+9x-9 = (2x^2+x-1)(x^2+9)$$

$$= (x+1)(2x-1)(x-3i)(x+3i)$$

$$x^2+9=0$$

$$x^2=-9$$

$$x=\pm\sqrt{-9}$$

$$x=\pm 3i$$

$$x=3i$$

$$x=-3i$$