Mth 146: Day 9
7.8 Improper Integrals: Type II

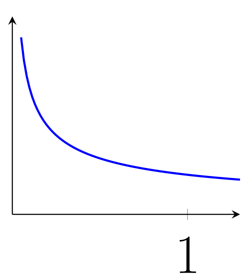
1 Part 1: New Concepts & Worked Examples

Recall

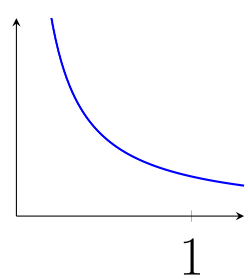
Integrating Unbounded Functions

The Fundamental Theorem of Calculus does not apply to integrands that have discontinuities such as vertical asymptotes, but we can again use limits to determine whether integrals of unbounded functions converge or diverge.

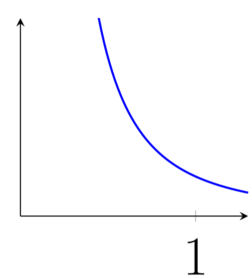
Consider the same three functions over the interval $(0, 1]$:



$\frac{1}{\sqrt{x}}$



$\frac{1}{x}$



$\frac{1}{x^2}$

Algebraically:

Example 1. $\displaystyle\int_0^1 \frac{1}{x^{2/3}}\,dx$

Suppose a and b are any real numbers and $f(x)$ is a function.

1. If f is continuous on $(a, b]$ but not at $x = a$, then

2. If f is continuous on $[a, b)$ but not at $x = b$, then

3. If f is continuous on $[a, c) \cup (c, b]$, then we can define the integral over $[a, b]$ as the sum of the first two cases

In all three cases, if the limits involved exist, then we say that the improper integral **converges** to the value determined by those limits, and if the limits are infinite or fail to exist, then we say that the improper integral **diverges**.

Example 2. Determine whether the improper integral converges and, if so, evaluate it.

$$\int_0^3 \frac{dx}{x-1}$$

Caution:

Example 3. Evaluate $\displaystyle\int_0^1 \ln x \, dx$.

Determine if the integral converges or diverges. Final answers are given as footnotes at the bottom of the page — only the final answer, not the steps, so use them to check your work rather than replace it.

1. $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx.$[1]

2. $\displaystyle\int_{1/2}^2 \frac{1}{x\ln(x)}\,dx.$[2]

[1] Answer: Converges to $2\sqrt{3}$.

[2] Answer: Diverges. (The discontinuity at $x = 1$, where $\ln x = 0$, sits inside the interval; $\int \frac{dx}{x \ln x} = \ln|\ln x| + C$, and $\ln|\ln x| \to -\infty$ as $x \to 1$.)

Challenge: On Day 8, we worked together to show that $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges exactly when $p > 1$.

Now it's your turn to do the same thing for a Type II improper integral. Using the same approach — write the integral as a limit, find the antiderivative for $p \neq 1$, and evaluate the limit — figure out for yourself exactly which values of p make

$$\int_0^1 \frac{1}{x^p}\,dx$$

converge, and which values make it diverge. If it converges, what does it converge to?

Once you've got a conjecture, think about *why* the condition on p comes out differently here than it did for the Type I integral on Day 8.

3 Part 3: Reflection

For each item below, rate your comfort level and write a reflection at the end. Then submit your reflection using the QR code.

1. **Can you tell the difference between a Type I and a Type II improper integral just by looking at it?**
   1 2 3 4 5

2. **How comfortable are you setting up Type I and Type II improper integrals as limits?**
   1 2 3 4 5

3. **How comfortable are you actually calculating those limits once the integral is set up?** 1 2 3 4 5

4. **Determine, for a Type I p-integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$, which values of p make it converge versus diverge.**
   1 2 3 4 5