Mth 146: Day 15
11.5 Alternating Series & Absolute Convergence
11.6 The Ratio and Root Tests

1 Part 1: New Concepts & Worked Examples

Recall: The Alternating Series Test

An **alternating series** is a series whose terms are alternately positive and negative. Its nth term has the form

$$a_n = (-1)^{n-1} b_n \qquad \text{or} \qquad a_n = (-1)^n b_n,$$

where $b_n > 0$ (in fact, $b_n = |a_n|$).

Alternating Series Test. If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - \cdots \qquad (b_n > 0)$$

satisfies the conditions

(i) _____ for all n

(ii) _____

then the series is _____.

Example 1. Show that the **alternating harmonic series**

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

is convergent.

The Alternating Series Estimation Theorem

Alternating Series Estimation Theorem. If $s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies

(i) $b_{n+1} \leq b_n$ and (ii) $\lim_{n\to\infty} b_n = 0$,

then

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

In words: the size of the error in using s_n to approximate s is no bigger than the first neglected term b_{n+1}

Note: This rule is valid *only* for alternating series that satisfy the conditions of the Alternating Series Test. It does not apply to other types of series.

Example 2. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{(-1)^0}{0!} + \frac{(-1)^1}{1!} + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \cdots + \frac{(-1)^6}{6!} + \cdots$$

$$S = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} + \cdots$$

$0.001 > 0.0001$

First check that it passes AST with $b_n = \frac{1}{n!}$

i) $\lim_{n\to\infty} \frac{1}{n!} = 0$ ii) $\frac{1}{(n+1)!} \leq \frac{1}{n!}$ says passes AST (converges)

We want the error to be less than 0.0005.

Notice $b_7 = \frac{1}{5040} = 0.0001$

so therefore we approximate S with S_6

$$S_6 = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.36805$$

//

Absolute Convergence and Conditional Convergence

Given any series $\sum a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute values of the terms of the original series.

Definition 1. A series $\sum a_n$ is called **absolutely convergent** if $\sum |a_n|$ is convergent.

Definition 2. A series $\sum a_n$ is called **conditionally convergent** if it is convergent ($\sum a_n$ is convergent) but not absolutely convergent; that is, if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Note: If $\sum a_n$ is a series with positive terms, then $|a_n| = a_n$, so absolute convergence is the same as convergence in this case.

Example 3. Show that the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ is absolutely convergent.

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p series with $p = 2 > 1$. Thus $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent.

a.c.

Example 4. Show that the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is conditionally convergent.

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic series diverges)

Since $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges by AST but $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right|$ diverges then $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conditionally convergent.

Theorem 3. If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Think of it this way: absolute convergence is a *stronger* type of convergence. An absolutely convergent series converges no matter what the signs of its terms are, while a conditionally convergent series (like the alternating harmonic series) would *not* converge if we made all of its negative terms positive.

The Ratio Test

The Ratio Test.

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the Ratio Test is inconclusive, that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Tip: The Ratio Test is usually conclusive when the nth term contains an **exponential** or a **factorial**.

Example 5. Test the series $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$ for absolute convergence.

a_n

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n-1}(n+1)^3}{3^{n-3}}}{\frac{(-1)^n n^3}{3^n}}\right| =$$

$$a_{n+1} = \frac{(-1)^{n+1}(n+1)^3}{3^{n+1}} = \lim_{n\to\infty}\frac{(n+1)^3}{3^{n-3}}\cdot\frac{3^n}{n^3} =$$

$$= \lim_{n\to\infty}\frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \lim_{n\to\infty}\frac{1}{3}\left(1+\frac{1}{n}\right)^3 = \frac{1}{3}$$

Since $1/3 < 1$ by Ratio Test a.c.

$L = 1/3 < 1$

Example 6. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} =$$

$$= \lim_{n\to\infty} \frac{(n+1)^n (n+1)}{n!\,(n+1)} \cdot \frac{n!}{n^n} =$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n =$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e = 2.7182 > 1$$

So by Ratio Test $e > 1$ so $\sum \frac{n^n}{n!}$ diverges.

Example 7. Apply the Ratio Test to each of the following series. What do you notice?

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

so RT is inconclusive but we know this diverges (harmonic series).

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$$

but this series converges (p series $p=2$)

Careful! When $L = 1$, the Ratio Test gives **no information**: the series could converge or diverge. The Ratio Test will *always* fail for p-series.

The Root Test.

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

Tip: The Root Test is convenient to apply when ***n*th powers** occur. Also, if $L = 1$ in the Ratio Test, don't try the Root Test (and vice versa) — L will be 1 again.

Example 8. Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$. a_n

Apply Root Test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n}$$

$$= \lim_{n\to\infty} \frac{2n+3}{3n+2} = 2/3 < 1 \text{ so by RT } \sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n \text{ is a.c.}$$

Example 9. Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^n$ converges or diverges.

Root Test: $\lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{n+1}\right)^n} = \lim_{n\to\infty}\frac{n}{n+1} =$

$= \lim_{n\to\infty}\frac{n/n}{\frac{n}{n}+1/n} = \lim_{n\to\infty}\frac{1}{1+1/n} = 1$

$L=1$ so inconclusive.

Try Ratio Test: $\lim_{n\to\infty}\dfrac{\left(\frac{n+1}{n+1+1}\right)^{n+1}}{\left(\frac{n}{n+2}\right)^n} = \lim_{n\to\infty}\dfrac{\left(\frac{n+1}{n+2}\right)^n\left(\frac{n+1}{n+2}\right)}{\left(\frac{n}{n+1}\right)^n}$

$= \lim_{n\to\infty}\left(\frac{n+1}{n+2}\cdot\frac{n+1}{n}\right)^n\left(\frac{n+1}{n+2}\right) = \cdots = 1$

Lesson: When testing a series for convergence or divergence, it is often helpful to apply the **Test for Divergence** first, before attempting other tests.

2 Part 2: Now You Try It

Determine whether each series is **absolutely convergent**, **conditionally convergent**, or **divergent**.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$

   Solution: $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^3}\right| = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p-series ($p = 3 > 1$), so the given series is **absolutely convergent**.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$

   Solution: First test for absolute convergence:
   $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt[3]{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ diverges (p-series with $p = \frac{1}{3} \leq 1$), so the series is *not* absolutely convergent.

   Now use the AST with $b_n = \dfrac{1}{\sqrt[3]{n}}$:

   $b_{n+1} = \dfrac{1}{\sqrt[3]{n+1}} < \dfrac{1}{\sqrt[3]{n}} = b_n$ and $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$, so the series converges.

   It converges but not absolutely, so it is **conditionally convergent**.

3. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n}{2n+1}$

Solution: This series is alternating, but $\dfrac{n}{2n+1}\to\dfrac{1}{2}$, so

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}(-1)^n\frac{n}{2n+1}$$

does not exist (the terms bounce between values near $\frac{1}{2}$ and $-\frac{1}{2}$). So the series is **divergent** by the Test for Divergence.

4. $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n!}$

Solution: Ratio Test with $a_n=\dfrac{2^n}{n!}$:

$$\left|\frac{a_{n+1}}{a_n}\right|=\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}=\frac{2}{n+1}\to 0<1.$$

So the series is **absolutely convergent**.

5. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n}{2^n n^3}$

Solution: Ratio Test with $a_n = (-1)^{n-1}\dfrac{3^n}{2^n n^3}$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}}{2^{n+1}(n+1)^3}\cdot\frac{2^n n^3}{3^n} = \frac{3}{2}\left(\frac{n}{n+1}\right)^3 \to \frac{3}{2} > 1.$$

So the series is **divergent** by the Ratio Test.

6. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$

Solution: Root Test:

$$\sqrt[n]{|a_n|} = \frac{n^2+1}{2n^2+1} = \frac{1+\frac{1}{n^2}}{2+\frac{1}{n^2}} \to \frac{1}{2} < 1.$$

So the series is **absolutely convergent**.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$

Solution: Root Test:

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{2^n}{n^n}} = \frac{2}{n} \to 0 < 1.$$

So the series is **absolutely convergent**.

3 Part 3: Reflection

For each item below, rate your comfort level and jot a quick note: What clicked? What's still fuzzy? What would help?

1. **Check the two conditions of the Alternating Series Test and use it to show an alternating series converges.** 1 2 3 4 5

2. **Use the Alternating Series Estimation Theorem to approximate the sum of a series to a given accuracy.** 1 2 3 4 5

3. **Classify a series as absolutely convergent, conditionally convergent, or divergent.** 1 2 3 4 5

4. **Apply the Ratio Test, including simplifying $|a_{n+1}/a_n|$ with factorials and exponentials.** 1 2 3 4 5

5. **Apply the Root Test, and recognize when the Ratio or Root Test is inconclusive ($L = 1$).** 1 2 3 4 5