

## 1. Sequences

In this section we are going to introduce Sequences and Series which are fundamental mathematical concepts that are important for understanding patterns and are used in practice to model population growth in biology and analyze algorithms in computer science.

**Definition.** Roughly speaking, a **sequence** is an infinite list of numbers with some order or pattern.

**Example 1.** A simple example is the sequence:

$$\begin{array}{ccccccc}
 \text{first term} \downarrow & & & & & & \\
 5, & 10, & 15, & 20, & 25, & \dots & 5n \\
 \hline 5 \cdot 1 & 5 \cdot 2 & 5 \cdot 3 & 5 \cdot 4 & \dots & & 5n \\
 \hline a_1 & a_2 & a_3 & a_4 & \dots & & a_n, \dots
 \end{array}$$

The numbers in the sequence are called terms and are often written as  $a_1, a_2, a_3, \dots$ . The dots mean that the list continues forever.

**Notation:**  
 $a_2 = 10$  (means the second term of the sequence)

$a_n$  = the general term of the sequence or the  $n^{\text{th}}$  term of the sequence

$\{a_n\}$  = represents the entire sequence  
 $\{a_n\} = \{a_1, a_2, a_3, \dots\}$   
 $\{a_n\} = \{5, 10, 15, \dots\}$

$n$  - is a positive integer  
 $1, 2, 3, 4, \dots$

We can describe the pattern of the sequence displayed above by a formula for the general term:

$$a_n = 5n$$

**Example 2.** Find the first three terms and the 100th term of the sequence defined by each formula

1.  $a_n = 2n - 1$

$$a_1 = 2 \cdot 1 - 1 = 1$$

$$a_2 = 2 \cdot 2 - 1 = 3$$

$$a_3 = 2 \cdot 3 - 1 = 5$$

$$a_{100} = 2 \cdot 100 - 1 = 199$$

$$\{a_n\} = \{1, 3, 5, \dots\}$$

2.  $b_n = n^2 - 1$

$$b_1 = 1^2 - 1 = 0$$

$$b_2 = 2^2 - 1 = 3$$

$$b_3 = 3^2 - 1 = 8$$

$$b_{100} = 100^2 - 1 = 9999$$

$$\{b_n\} = \{0, 3, 8, \dots\}$$

3.  $c_n = \frac{n}{n+1}$

$$c_1 = \frac{1}{1+1} = \frac{1}{2}$$

$$c_2 = \frac{2}{2+1} = \frac{2}{3}; \quad c_3 = \frac{3}{3+1} = \frac{3}{4}$$

$$c_{100} = \frac{100}{100+1} = \frac{100}{101}$$

$$\{c_n\} = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots \right\}$$

4.  $d_n = \frac{(-1)^{n+1}}{n}$

$$d_4 = \frac{(-1)^{4+1}}{4} = -\frac{1}{4}$$

$$d_1 = \frac{(-1)^{1+1}}{1} = \frac{(-1)^2}{1} = 1$$

$$d_2 = \frac{(-1)^{2+1}}{2} = \frac{(-1)^3}{2} = -\frac{1}{2}$$

$$d_3 = \frac{(-1)^{3+1}}{3} = \frac{(-1)^4}{3} = \frac{1}{3}$$

$$d_{100} = \frac{(-1)^{100+1}}{100} = -\frac{1}{100}$$

$$\{d_n\} = \left\{ 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots \right\}$$

**Example 3.** Find the general term of a sequence whose first several terms are given.

(a)  $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots$

$$n=1, \quad a_1 = \frac{2}{3} = \frac{1+1}{1+2}$$

$$n=2, \quad a_2 = \frac{3}{4} = \frac{2+1}{2+2}$$

$$n=3, \quad a_3 = \frac{4}{5} = \frac{3+1}{3+2}$$

$$n=4, \quad a_4 = \frac{5}{6} = \frac{4+1}{4+2}$$

$$n, \quad a_n = ? \quad a_n = \frac{n+1}{n+2}$$

(b)  $-2, 4, -8, 16, -32, \dots$

$$n=1, \quad b_1 = -2 = 2^1 (-1)^1$$

$$n=2, \quad b_2 = 4 = 2^2 (-1)^2$$

$$n=3, \quad b_3 = -8 = 2^3 (-1)^3$$

$$n=4, \quad b_4 = 16 = 2^4 (-1)^4$$

$$n=5, \quad b_5 = -32 = 2^5 (-1)^5$$

$$n, \quad b_n = ?$$

$$b_n = (-1)^n 2^n$$

**Definition.** An alternating sequence is a sequence with terms that alternate in sign.

$- , + , - , + , - , + , \dots \rightarrow$  use  $(-1)^n$  in the  $n^{\text{th}}$  term formula  
 $+ , - , + , - , + , - , \dots \rightarrow$  use  $(-1)^{n+1}$  in the  $n^{\text{th}}$  term formula

## 2. Recursively defined sequences

Some sequences do not have simple defining formulas like those of the preceding example.

**Definition.** When the  $n^{\text{th}}$  term of a sequence may depend on some or all of the terms preceding it we call the sequence recursive sequence.

**Example 4.** Example of recursive sequence:

$a_1 = 4$  and  $a_n = 2a_{n-1} + 1$  for  $n > 1$ .

next term

$a_1, a_2, \dots, a_{n-1}, a_n, \dots$   
 $\downarrow$   
 $4$

twice the previous term plus 1

$4, 2 \cdot 4 + 1, \dots$

$4, 9, 2 \cdot 9 + 1, \dots$

$4, 9, 19, \dots$

$$\begin{aligned} a_1 &= 4 \\ a_2 &= 2a_1 + 1 = 9 \quad (n=2 \text{ in formula}) \\ a_3 &= 2a_2 + 1 = 2 \cdot 9 + 1 = 19 \\ &\vdots \end{aligned}$$

**Example 5.** The Fibonacci Sequence:

Find the first 7 terms of the sequence defined recursively by

$a_1 = 1, a_2 = 1$  and  $a_n = a_{n-1} + a_{n-2}$ .

$\downarrow$  next term  $\downarrow$  previous term  $+$  term before previous term

$$a_1 = 1$$

$$a_2 = 1$$

$$a_3 = a_2 + a_1 = 1 + 1 = 2$$

$$a_4 = a_3 + a_2 = 2 + 1 = 3$$

$$a_5 = a_4 + a_3 = 3 + 2 = 5$$

$$\{a_n\} = \{1, 1, 2, 3, 5, 8, 13, \dots\}$$

### 3. Factorial notation

Factorials form the basis for important series like Taylor series expansions.

**Definition.** The factorial of a positive integer  $n$ , written as  $n!$ , is the product of all positive integers less than or equal to  $n$ .  
For example,  $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$   
 $5! = 120$

$$10! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = 3628800$$

calc.: press 10  $\rightarrow$  MATH  $\rightarrow$  PRB  $\rightarrow$  !

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n$$

By definition,  $0! = 1$

**Example 6.** Evaluate the following expressions.

$$\begin{aligned} \text{(a)} \quad \frac{8!}{3! \cdot 5!} &= \frac{\cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot 6 \cdot 7 \cdot 8}{2! \cdot \cancel{5!}} \\ &= \frac{6 \cdot 7 \cdot 8}{2!} = \frac{6 \cdot 7 \cdot 8}{1 \cdot 2} \\ &= 56 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{n!}{(n+2)!} &= \frac{\cancel{n!}}{\cancel{1 \cdot 2 \cdot 3 \cdots n} \cdot (n+1)(n+2)} \\ &= \frac{1}{(n+1)(n+2)} \end{aligned}$$

**Example 7.** Given the sequence defined by  $b_n = \frac{n^2}{(n+1)!}$  find  $b_1$  and  $b_6$ .

$$b_1 = \frac{1^2}{(1+1)!} = \frac{1}{2!} = \frac{1}{1 \cdot 2} = \frac{1}{2}$$

$$b_6 = \frac{6^2}{(6+1)!} = \frac{36}{7!} = \frac{36}{5040} = \frac{1}{140}$$