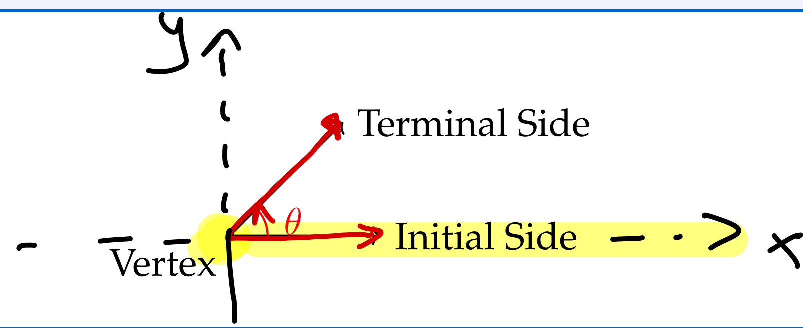
4.1 Radians and Degrees Measure and 4.2 Trig functions: The Unit Circle

1 Angles

1.1 Basic Definitions

Definition : Angle

An **angle** is formed by two rays that share a common endpoint. The common endpoint is called the **vertex** of the angle, one ray is called the **initial side**, and the other ray is called the **terminal side**.



Definition : Standard Position

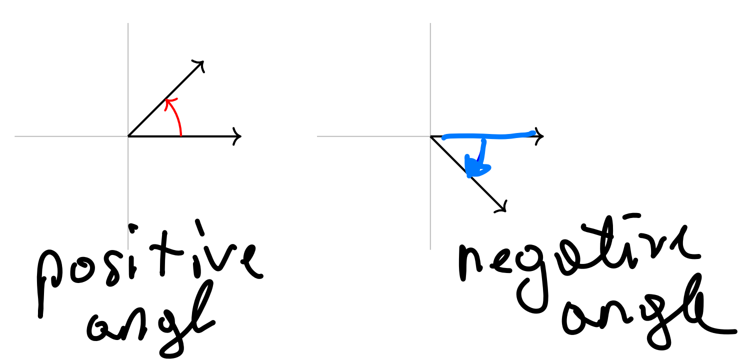
An angle is in **standard position** if:

- The vertex is at the origin of a rectangular coordinate system
- The initial side is along the positive x-axis

Notation: we use Greek letters to denote angles $\theta, \alpha, \beta, \gamma$

Definition : Positive and Negative Angles

- A **positive angle** is generated by a counterclockwise rotation from the initial side to the terminal side.
- A **negative angle** is generated by a clockwise rotation from the initial side to the terminal side.



Definition : Coterminal Angles

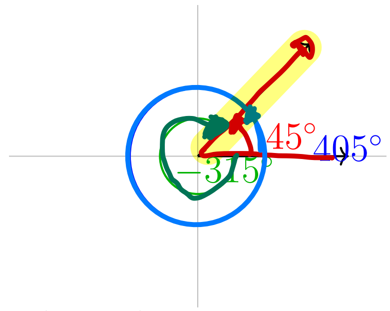
Coterminal angles are angles in standard position that have the same terminal side.

If θ is an angle, then $\theta + 360°n$ where n is an integer, is coterminal with θ.

$\theta - 360°$

$\theta + 2(360°)$

$\theta - 2(360°)$



$$\begin{array}{r} 360° + \\ 45 \\ \hline 405 \end{array}$$

Coterminal Angles: $45°$, $405°$, and $-315°$

Example 1: Finding Coterminal Angles

Find two coterminal angles (one positive and one negative) for $\theta = 60°$.

Solution:

For a positive coterminal angle, add $360°$: $60° + 360° = 420°$

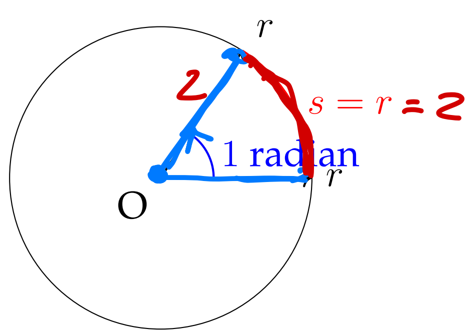
For a negative coterminal angle, subtract $360°$: $60° - 360° = -300°$

2 Radian Measure

2.1 Definition of Radian

Definition : Radian

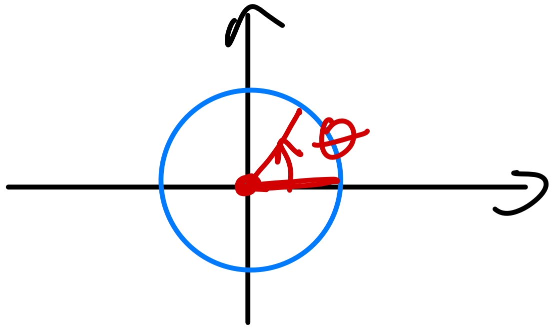
A **radian** is the measure of an angle that, when placed at the center of a circle, intercepts an arc equal in length to the radius of the circle.



S = arc portion of the circle

Definition : Central Angle

A **central angle** is an angle whose vertex is at the center of a circle, with its sides being radii of the circle.



θ is a central angle

$$\boxed{360^\circ = 2\pi} \text{ (in radians)}$$

$/360$

$1^\circ = \frac{2\pi}{360} \Rightarrow 1^\circ = \frac{\pi}{180}$

3 Degree and Radian Conversions

3.1 Converting Between Degrees and Radians

Definition : Conversion Formulas

To convert between degrees and radians, use the following relationships:

$$\boxed{1 \text{ radian} = \frac{180^\circ}{\pi}} \approx 57.3^\circ$$

$$\boxed{1^\circ = \frac{\pi}{180} \text{ radians}} \approx 0.01745 \text{ radians}$$

Therefore:

$$\text{radians} = \text{degrees} \times \frac{\pi}{180}$$

$$\text{degrees} = \text{radians} \times \frac{180}{\pi}$$

Example 2: Converting from Degrees to Radians

Convert the following angles from degrees to radians in terms of π: a) 45° b) 120° c) -30°

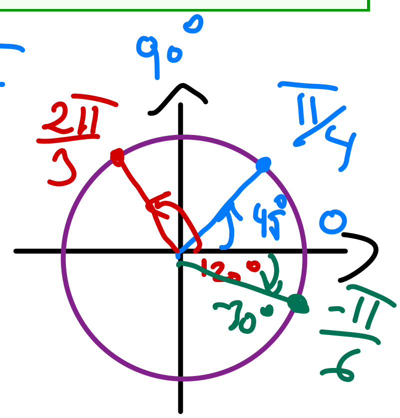
multiply by $\pi/180^\circ$

Solution:

a) 45° in radians: $45^\circ \cdot \frac{\pi}{180^\circ} = \frac{45\pi}{180} = \frac{\pi}{4}$

b) 120° in radians: $120^\circ \cdot \frac{\pi}{180^\circ} = \frac{120\pi}{180} = \frac{2\pi}{3}$

c) -30° in radians: $-30^\circ \cdot \frac{\pi}{180^\circ} = -\frac{\pi}{6}$



Example 3: Converting from Radians to Degrees

Convert the following angles from radians to degrees: a) $\frac{\pi}{3}$ b) $\frac{5\pi}{4}$ c) $-\frac{\pi}{2}$

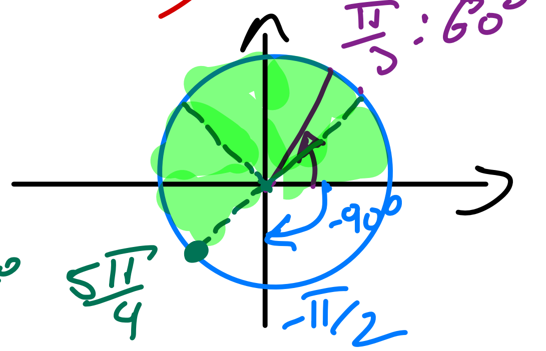
multiply by $\frac{180^\circ}{\pi}$

Solution:

a) $\frac{\pi}{3}$ in degrees: $\frac{\cancel{\pi}}{3} \cdot \frac{180^\circ}{\cancel{\pi}} = 60^\circ$

b) $\frac{5\pi}{4}$ in degrees: $\frac{5\cancel{\pi}}{\cancel{4}} \cdot \frac{\overset{45}{\cancel{180}}{}^\circ}{\cancel{\pi}} = 5 \cdot 45^\circ = 225^\circ$

c) $-\frac{\pi}{2}$ in degrees: $-\frac{\cancel{\pi}}{2} \cdot \frac{180^\circ}{\cancel{\pi}} = -90^\circ$



4 Complementary, Supplementary Angles, and Arc Length

4.1 Complementary and Supplementary Angles

Definition : Complementary Angles

Two positive angles are **complementary** if their sum equals 90° or $\frac{\pi}{2}$ radians.
If angles A and B are complementary, then:

$$A + B = 90^\circ \text{or } A + B = \frac{\pi}{2} \text{ radians}$$

The **complement** of angle A is $(90^\circ - A)$ or $(\frac{\pi}{2} - A)$ in radians.

Definition : Supplementary Angles

Two positive angles are **supplementary** if their sum equals 180° or π radians.
If angles A and B are supplementary, then:

$$A + B = 180^\circ \text{or } A + B = \pi \text{ radians}$$

The **supplement** of angle A is $(180^\circ - A)$ or $(\pi - A)$ in radians.

Example 4: Finding Complements and Supplements

Find the complement and supplement of each angle: a) $25°$, b) $\frac{\pi}{4}$ radians, and c) $100°$

Solution:

a) For $\theta = 25°$:

$$\text{Complement} = 90° - 25° = 65°$$
$$\text{Supplement} = 180° - 25° = 155°$$

b) For $\theta = \frac{\pi}{4}$ radians:

$90° \to \pi/2$

$180° \to \pi$

$$\text{Complement} = \pi/2 - \pi/4 = \pi/4$$
$$\text{Supplement} = \pi - \pi/4 = \frac{4\pi}{4} - \frac{\pi}{4} = 3\pi/4$$

c) For $\theta = 100°$:

$$\text{Complement} = \text{no complement}$$
$$\text{Supplement} = 180° - 100 = 80°$$

4.2 Arc Length and Area of a Sector

Definition : Arc Length and Area of a sector

If a central angle θ in a circle of radius r intercepts an arc, then the arc length s is given by:

$$s = r\theta \quad *$$

where θ must be measured in radians. The area A of a sector is given by:

$$A = \frac{1}{2}r^2\theta \quad *$$

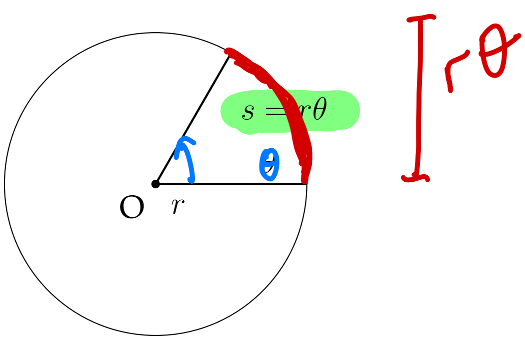


Figure 1: Arc length of a sector

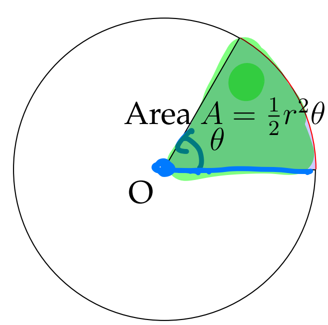


Figure 2: Area of a sector

Example 5: Finding Arc Length

Find the arc length intercepted by a central angle of $\frac{\pi}{3}$ radians in a circle with radius 12 cm.

Solution: Using the arc length formula with $r = 12$ cm and $\boxed{\theta = \frac{\pi}{3}}$ radians:

$$S = r\theta = 12\left(\pi/3\right) = 4\pi \text{ cm} \approx 12.57 \text{ cm}$$

Example 6: Finding Area of a Sector

Find the area of a sector formed by a central angle of 72° in a circle with radius 5 m.

$$A = \frac{1}{2} r^2 \theta$$

$$\theta = 72^\circ \cdot \frac{\pi}{180^\circ} = \frac{72\pi}{180} = \frac{2\pi}{5}$$

$$A = \frac{1}{2}(5m)^2 \frac{2\pi}{5} = 5\pi \text{ m}^2 \approx 15.71 \text{ m}^2$$

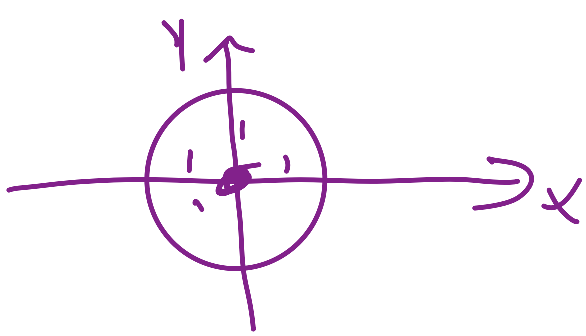
5 The Unit Circle and Trigonometric Functions

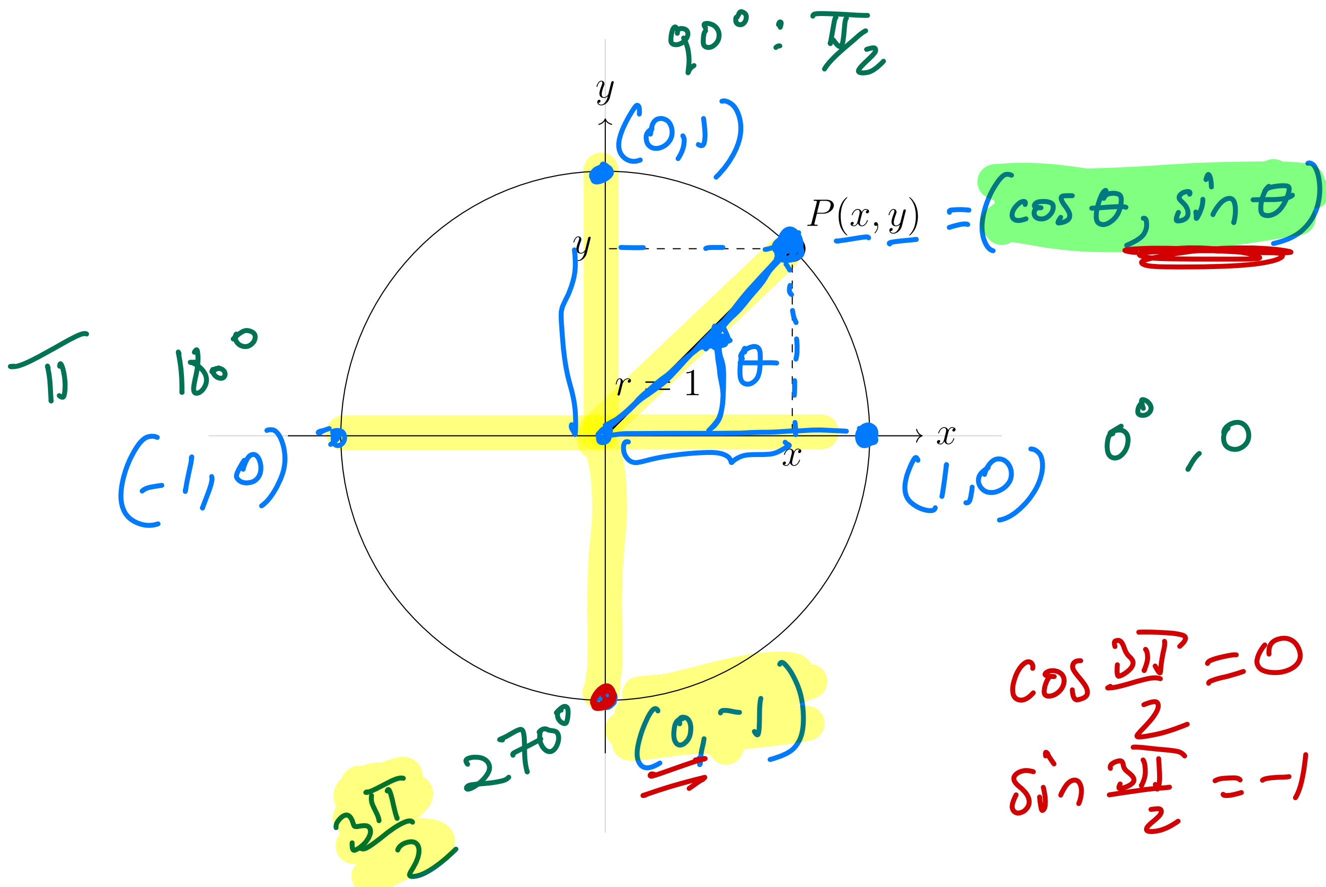
5.1 Definition of the Unit Circle

Definition : Unit Circle

The **unit circle** is a circle with radius 1 and center at the origin of the coordinate plane. Every point on the unit circle is at a distance of 1 unit from the origin.
The equation of the unit circle is:

$$x^2 + y^2 = 1$$





5.2 Six Trigonometric Functions: sine, cosine, tangent, cosecant, secant, cotangent

sin cos tan csc sec cot

Definition : Primary Trigonometric Functions

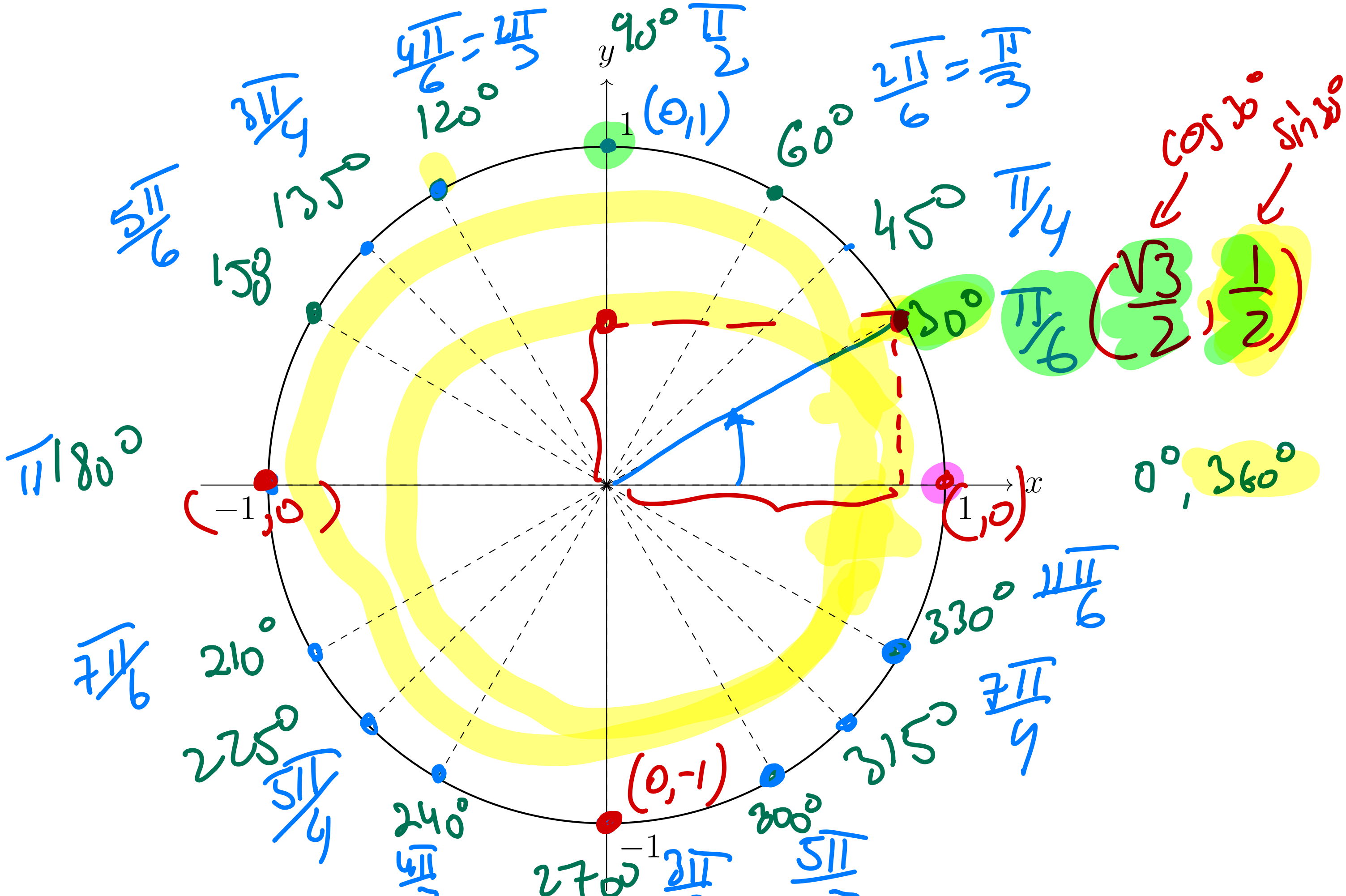
For a point $P(x, y)$ on the unit circle corresponding to an angle θ in standard position:

$$\sin\theta = y$$
$$\cos\theta = x$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x} \quad \cos\theta \neq 0$$

Definition : Reciprocal Trigonometric Functions

The reciprocal trigonometric functions are defined as:

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{y}, \quad \sin\theta \neq 0$$
$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{x}, \quad \cos\theta \neq 0$$
$$\cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}, \quad \sin\theta \neq 0$$



Unit Circle with Common Angles

5.3 Evaluating Trigonometric Functions

Example 7: Evaluating Trig Functions for Special Angles

Find the values of all six trigonometric functions for $\theta = 30^\circ$.

Solution:

For $\theta = 30^\circ = \frac{\pi}{6}$ radians, the point on the unit circle is:

$$P(\cos 30^\circ, \sin 30^\circ) = P\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

Therefore:

$\sin 30^\circ =$ $1/2$

$\cos 30^\circ =$ $\sqrt{3}/2$

$\tan 30^\circ =$ $y/x = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{3}} = \frac{1}{\sqrt{3}} \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\csc 30^\circ =$ $\frac{1}{\sin 30^\circ} = \frac{1}{1/2} = 2$

$\sec 30^\circ =$ $\frac{1}{\cos 30^\circ} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}} \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

$\cot 30^\circ =$

$\frac{\sqrt{3}}{2} \div \frac{1}{2} = \sqrt{3}$

Example 8: Finding Trig Functions from a Point

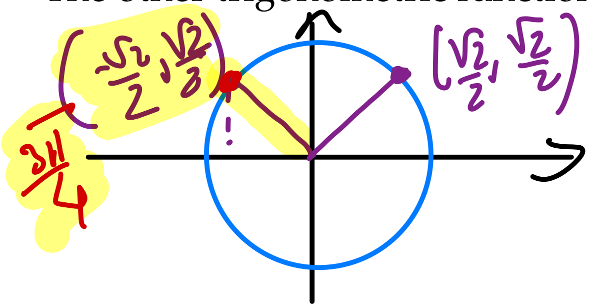
If $P(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ is a point on the unit circle corresponding to angle θ, find all six trigonometric functions of θ.

Solution:
Since the point is on the unit circle, we know that:

$P(\cos\theta, \sin\theta)$
$P(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

$$\cos\theta = -\sqrt{2}/2$$
$$\sin\theta = \sqrt{2}/2$$

The other trigonometric functions can be found using these values:



$$\tan\theta = \sin\theta/\cos\theta = \frac{\sqrt{2}}{2} \div -\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} \cdot \frac{-2}{\sqrt{2}} = -1$$
$$\csc\theta = 1/\sin\theta = 1/\sqrt{2}/2 = \frac{2}{\sqrt{2}}$$
$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{-\sqrt{2}/2} = -2/\sqrt{2}$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta} = -\frac{\sqrt{2}}{2} \div \frac{\sqrt{2}}{2} = -1$$

5.4 Domains of Trigonometric Functions

Definition : Domains of Trigonometric Functions

The domains of the six trigonometric functions are:

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\sin\theta$: All real numbers

$\cos\theta$: All real numbers

odd multiples of $\pi/2$

$\tan\theta$: All real numbers except $\theta = \frac{\pi}{2} + n\pi$ (where n is an integer)

$\frac{1}{\sin\theta}$ $\csc\theta$: All real numbers except $\boxed{\theta = n\pi \text{ (where } n \text{ is an integer)}}$ multiples of π

$\sec\theta$: All real numbers except $\theta = \frac{\pi}{2} + n\pi$ (where n is an integer)

$\frac{\cos\theta}{\sin\theta}$ $\cot\theta$: All real numbers except $\boxed{\theta = n\pi}$ (where n is an integer)

5.5 Even and Odd Trigonometric Functions

Definition : Even and Odd Functions

A function f is **even** if $f(-x) = f(x)$ for all x in the domain. A function f is **odd** if $f(-x) = -f(x)$ for all x in the domain.

Note : Even and Odd Trigonometric Functions

- $\sin\theta$, $\csc\theta$, $\tan\theta$ $\cot\theta$ are odd
- $\cos\theta$ and $\sec\theta$ are even.

5.6 Periodicity of Trigonometric Functions

values start repeating

Definition : Periodic Functions

A function f is **periodic** if there exists a positive real number p such that $f(x+p) = f(x)$ for all x in the domain. The smallest such value of p is called the **period** of the function.

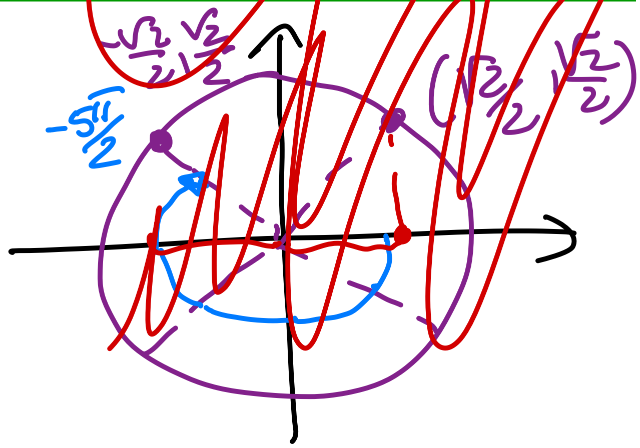
Note : Periods of Trigonometric Functions

- $\sin\theta$, $\cos\theta$, $\sec\theta$, $\csc\theta$ have a period of 2π.
- $\tan\theta$ and $\cot\theta$ have a period of π.

Example 9: Using Even/Odd Properties

Find the exact value of $\cos(-5\pi/4)$ without using a calculator.

Solution:



6 Summary of Key Concepts

1. **Angles** are formed by two rays with a common vertex; they have an initial side and a terminal side.
2. **Standard position** means the angle's vertex is at the origin and the initial side is along the positive x-axis.
3. **Positive angles** rotate counterclockwise; **negative angles** rotate clockwise.
4. **Coterminal angles** share the same terminal side and differ by multiples of $360°$ or 2π radians.
5. **Radians** and **degrees** are two common angle measures: π radians $= 180°$.
6. **Complementary angles** sum to $90°$ or $\frac{\pi}{2}$ radians.
7. **Supplementary angles** sum to $180°$ or π radians.
8. **Arc length** formula: $s = r\theta$ (where θ is in radians)
9. **Area of a sector** formula: $A = \frac{1}{2}r^2\theta$ (where θ is in radians)
10. **Unit circle** has radius 1 and is centered at the origin. Points on the unit circle are $(x, y) = (\cos\theta, \sin\theta)$.
11. **Six trigonometric functions**: sine, cosine, tangent, cosecant, secant, and cotangent.
12. **Even functions**: $\cos\theta$ and $\sec\theta$
13. **Odd functions**: $\sin\theta$, $\csc\theta$, $\tan\theta$, and $\cot\theta$
14. **Periodicity**: $\sin\theta$, $\cos\theta$, $\csc\theta$, and $\sec\theta$ have period 2π; $\tan\theta$ and $\cot\theta$ have period π.