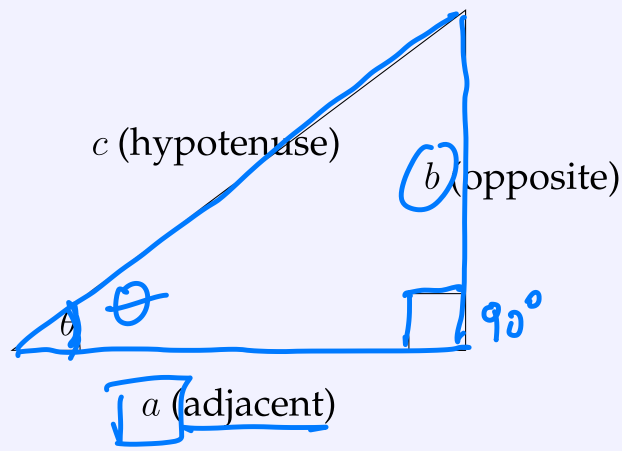
## Right Triangle Trigonometry

**Definition 4.3.1**

In a right triangle with acute angle $\theta$:
~~latex~~




For right triangles, the Pythagorean Theorem applies: $c^2 = a^2 + b^2$, where $c$ is the hypotenuse.

S=O/H C=A/H T=O/A

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{a}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{c}{b}$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{c}{a}$$

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{a}{b}$$

## Fundamental Identities

### Reciprocal Identities

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$
$$\sin(\theta) = \frac{1}{\csc(\theta)}$$
$$\sec(\theta) = \frac{1}{\cos(\theta)}$$
$$\cos(\theta) = \frac{1}{\sec(\theta)}$$
$$\cot(\theta) = \frac{1}{\tan(\theta)}$$
$$\tan(\theta) = \frac{1}{\cot(\theta)}$$

### Quotient Identities

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$
$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

$f(-x) = -f(x)$

$f(-x) = f(x)$

### Pythagorean Identities

$$\cos^2(\theta) + \sin^2(\theta) = 1$$
$$1 + \tan^2(\theta) = \sec^2(\theta)$$
$$\boxed{\cot^2(\theta) + 1 = \csc^2(\theta)}$$

### Even-Odd Properties

$\sin(-\theta) = -\sin(\theta)$ (odd)

$\cos(-\theta) = \cos(\theta)$ (even)

$\tan(-\theta) = -\tan(\theta)$ (odd)

$\csc(-\theta) = -\csc(\theta)$ (odd)

$\sec(-\theta) = \sec(\theta)$ (even)

$\cot(-\theta) = -\cot(\theta)$ (odd)

Pythagorean identities Remark:

$$\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

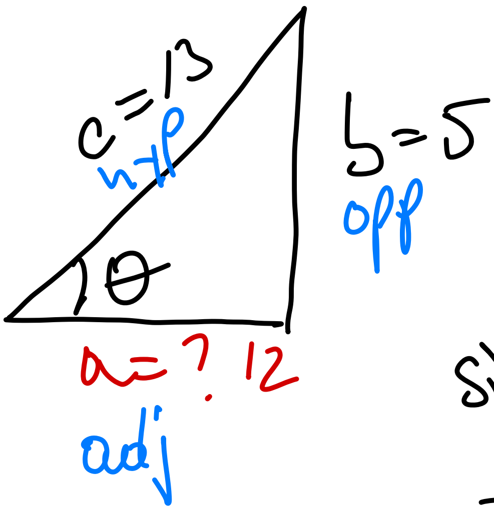
$$1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\boxed{1 + \cot^2\theta = \csc^2\theta}$$

**Example 1.** In the following examples find the exact values of the six trigonometric functions of the acute angle $\theta$ shown in the figure. Use the Pythagorean Theorem to determine the third side of the triangle.

**Example Problem 1**

Given: $c = 13$, $b = 5$.




$a^2+b^2=c^2$

$a^2=c^2-b^2=13^2-5^2=169-25$

$a^2=144 \quad a=\pm\sqrt{144}$

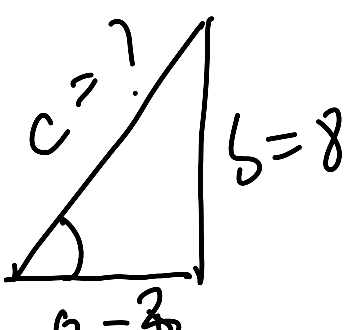
$a=12$

$\sin\theta=\frac{o}{h}=\frac{5}{13}; \quad \cos\theta=\frac{adj}{h}=\frac{12}{13}$

$\tan\theta=\frac{o}{adj}=\frac{5}{12}; \ldots$

**Example Problem 2**

Given: $a = 3$, $b = 8$.




$c^2=a^2+b^2=9+64=73$

$c=\sqrt{73}$

$\sin\theta=\frac{o}{h}=\frac{8}{\sqrt{73}}; \quad \cos\theta=\frac{adj}{h}=\frac{3}{\sqrt{73}}, \quad \tan=\frac{8}{3}$

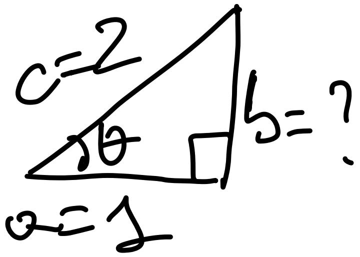
$\csc\theta=\frac{h}{o}=\frac{\sqrt{73}}{8}, \quad \sec\theta=\frac{h}{a}=\frac{\sqrt{73}}{3}$

$\cot\theta=\frac{3}{8}$

**Example 2.** Sketch a right triangle corresponding to the trigonometric function of the acute angle $\theta$. Use the Pythagorean Theorem to determine the third side of the triangle and then find the values of the other five trigonometric functions of $\theta$.

### Example Problem 1

Given: $\sec(\theta) = 2$




$\sec\theta = 2 = \frac{1}{\cos\theta} = \frac{hyp}{adj} = \frac{2}{1}$

$a^2+b^2=c^2$

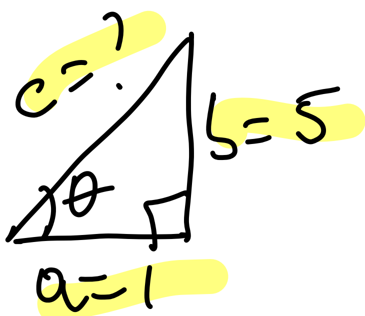
$1+b^2=2^2 \Rightarrow b^2 = 4-1 = 3$

$b=\sqrt{3}$

$\sin\theta = \frac{\sqrt{3}}{2}$, $\cos\theta = \frac{1}{2}$, $\tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3}$

### Example Problem 2

Given: $\tan(\theta) = 5$




$\tan\theta = \frac{o}{a} = \frac{5}{1}$

$c^2 = 1+25 = 26$

$c = \sqrt{26}$

$\sin\theta$ - . . .

## Cofunction Identities

Cofunctions of complementary angles (sum to $90^\circ or \frac{\pi}{2}$ radians) are equal.

cofunction

$$\sin(\theta) = \cos(90^\circ - \theta) \qquad \cos(\theta) = \sin(90^\circ - \theta)$$
$$\tan(\theta) = \cot(90^\circ - \theta) \qquad \cot(\theta) = \tan(90^\circ - \theta)$$
$$\sec(\theta) = \csc(90^\circ - \theta) \qquad \csc(\theta) = \sec(90^\circ - \theta)$$

**Example 3.** Use the function value(s) and the trigonometric identities to evaluate each trigonometric function.

### Example Problem 1

Given: $\sin(30^\circ) = \frac{1}{2}$, $\cos(30^\circ) = \frac{\sqrt{3}}{2}$.

(a) $\tan(30^\circ) =$ $\sin\theta/\cos\theta = 1/2 \div \frac{\sqrt{3}}{2} = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{3}} = \frac{1}{\sqrt{3}} \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

(b) $\sin(60^\circ) =$ $\cos(30^\circ) = \sqrt{3}/2$

(c) $\cos(60^\circ) =$ $\sin(30^\circ) = 1/2$

(d) $\cot(30^\circ) =$ $\frac{1}{\tan 30^\circ} = \sqrt{3}$

### Example Problem 2

Given: $\csc(\theta) = 5$, $\sec(\theta) = \frac{5\sqrt{6}}{12}$

$\csc\theta = \frac{1}{\sin\theta} \Rightarrow \sin\theta = \frac{1}{\csc\theta}$

(a) $\sin(\theta) =$ $1/\csc\theta = 1/5$

(b) $\cos(\theta) =$ $\frac{1}{\sec\theta} = \frac{1}{5\sqrt{6}/12} = \frac{12}{5\sqrt{6}} \frac{\sqrt{6}}{\sqrt{6}} = \frac{\cancel{12}^{2}\sqrt{6}}{5 \cdot \cancel{6}} = \frac{2\sqrt{6}}{5}$

(c) $\tan(\theta) =$ $\sin\theta/\cos\theta = 1/5 \div \frac{2\sqrt{6}}{5} = \frac{1}{5} \cdot \frac{5}{2\sqrt{6}} = \frac{5}{10\sqrt{6}} \dots$

(d) $\sec(90^\circ - \theta) =$ $\csc\theta = 5$

complementary $\quad 90^\circ - \theta + \theta = 90^\circ$

### Example Problem 3

Given: $\cot(\theta) = 3$

$\cot^2\theta + 1 = \csc^2\theta$

$3^2 + 1 = \csc^2\theta \Rightarrow \csc^2\theta = 10$

a) $\tan\theta = \frac{1}{\cot\theta} = \frac{1}{3}$

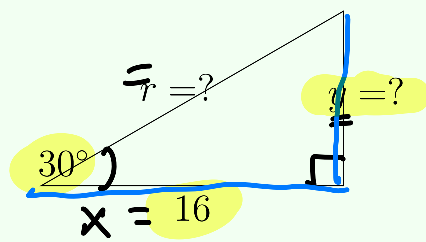
b) $\csc\theta = \sqrt{10}$

c) $\tan(90^\circ - \theta) = \cot(\theta) = 3$

**Example 4.** Find the exact values of the indicated variables.

**Example Problem 1**

Find $y$ and $r$ in a right triangle with angle $30°$, adjacent side $= 16$




Find $y$:

$\tan 30° = ?$

$\sin 30° = 1/2$

$\cos 30° = \sqrt{3}/2$

$$\tan 30° = \frac{\sin 30°}{\cos 30°} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{3}}$$

$$\tan 30° = \frac{1}{\sqrt{3}}$$

$$\tan 30° = \frac{\sqrt{3}}{3}$$

$$\frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}}{3} \Rightarrow \frac{y}{16} \times \frac{\sqrt{3}}{3} \Rightarrow 3y = 16\sqrt{3} \Rightarrow y = \frac{16\sqrt{3}}{3}$$

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Find $r$:

$\sin 30° = 1/2$

$$\frac{\text{opp}}{\text{hyp}} = \boxed{\frac{16\sqrt{3}/3}{r} = \frac{1}{2}}$$

$$\frac{2 \cdot 16\sqrt{3}}{3} = r$$

$$r = \frac{32\sqrt{3}}{3}$$