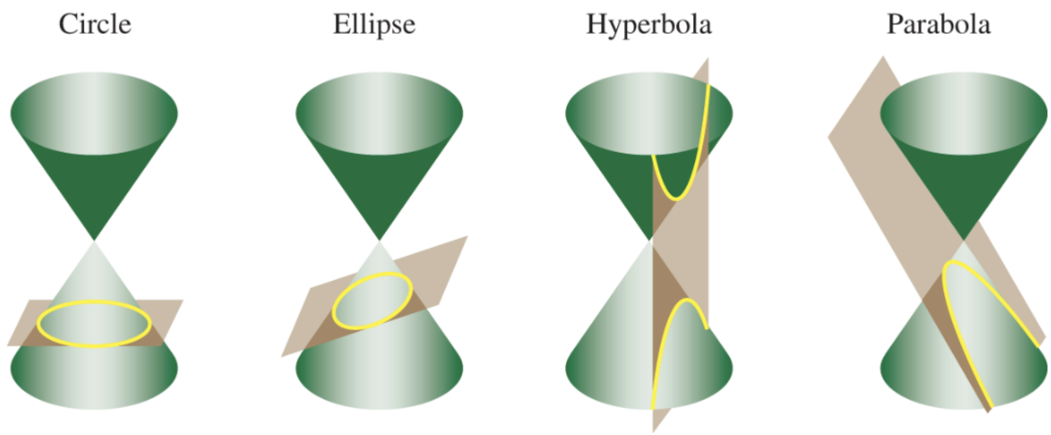
1. Ellipses

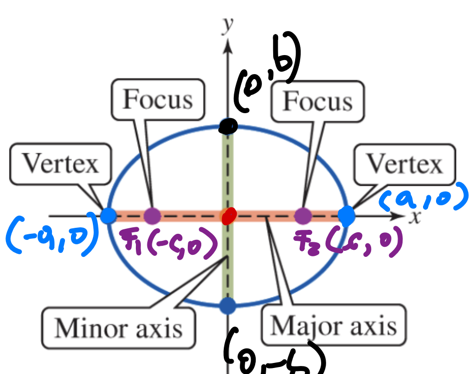
In this section we study ellipses from a geometric point of view.



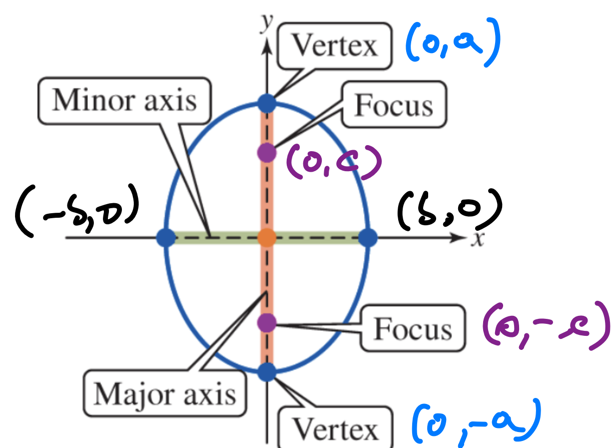
Definition: An ellipse is the set of all points in the plane such that the sum of the distance between a point (x, y) and two fixed points is a constant.

The fixed points are called the foci (plural of focus) of the ellipse.

An ellipse may be elongated in any direction. We will study only those that are elongated horizontally and vertically.



center : (0,0)
elongated horizontally (along the x axis)

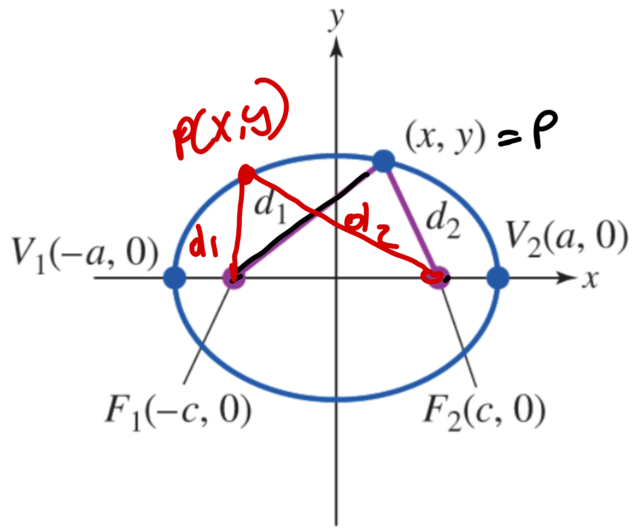


elongated vertically (along the y axis)

Length of major axis : $\boxed{2a}$
Length of minor axis : $2b$ where $b^2 = a^2 - c^2$

2. The equation of an ellipse

To obtain the simplest equation for an ellipse, we place the foci on the x-axis at $F_1 = (-c, 0)$ and $F_2 = (c, 0)$ so that the origin is halfway between them.



$d(P, F_1) + d(P, F_2) = \text{constant}$

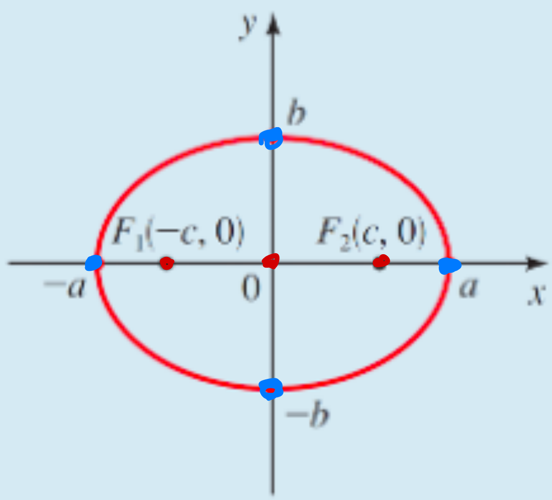
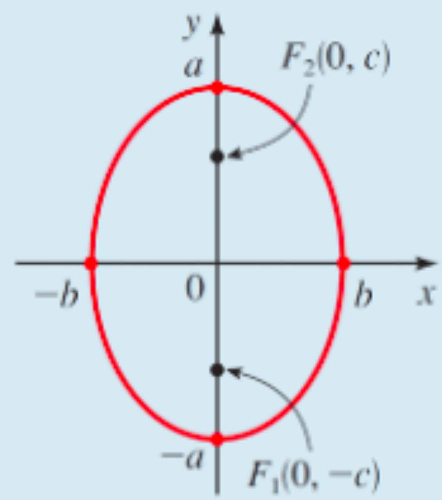
$d_1 + d_2 = \text{constant}$

3. Equations and Graphs of ellipses

The following box summarizes the equation and features of an ellipse centered at the origin.

ELLIPSE WITH CENTER AT THE ORIGIN

The graph of each of the following equations is an ellipse with center at the origin and having the given properties.

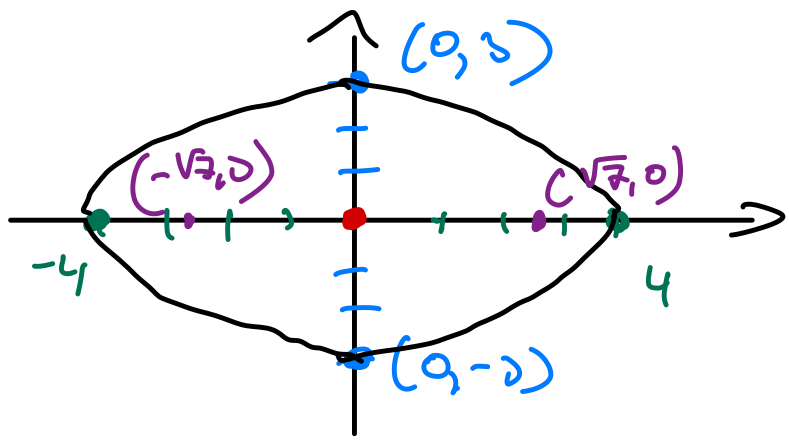
EQUATION	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $a > b > 0$	$\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$ $a > b > 0$
VERTICES	$(\pm a, 0)$	$(0, \pm a)$
MAJOR AXIS	Horizontal, length $2a$	Vertical, length $2a$
MINOR AXIS	Vertical, length $2b$	Horizontal, length $2b$
FOCI	$(\pm c, 0)$, $c^2 = a^2 - b^2$	$(0, \pm c)$, $c^2 = a^2 - b^2$
GRAPH		

Example 1: Given $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$, graph the ellipse and identify the center, foci and vertices.

Rewrite it in standard form: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\dfrac{x^2}{4^2} + \dfrac{y^2}{3^2} = 1$; $a = 4$, $b = 3$

Since $4 > 3$ we have an ellipse elongated horizontally (along x axis)



Center : $(0,0)$

vertices: $(a,0)$, $(-a,0)$ since $a=4$ vertice $(4,0)$, $(-4,0)$

since $b = 3 \Rightarrow$ minor axis is from $(0,-3)$ to $(0,3)$

Foci: $(c,0)$, $(-c,0)$ $c = ?$

$c^2 = a^2 - b^2 = 4^2 - 3^2 = 16 - 9 = 7$

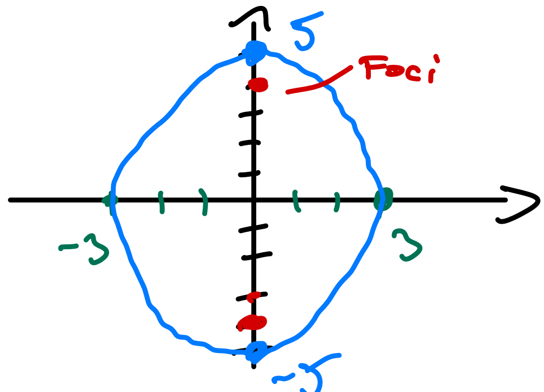
$c = \pm\sqrt{7} = \pm 2.65$

Example 2: Find the foci of the ellipse $25x^2 + 9y^2 = 225$, and sketch its graph.

We must bring it in standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Divide both sides by 225.

$$25x^2 + 9y^2 = 225 \quad \Big| \div 225$$



$$\frac{\cancel{25}x^2}{\cancel{225}\,9} + \frac{\cancel{9}y^2}{\cancel{225}\,25} = 1$$

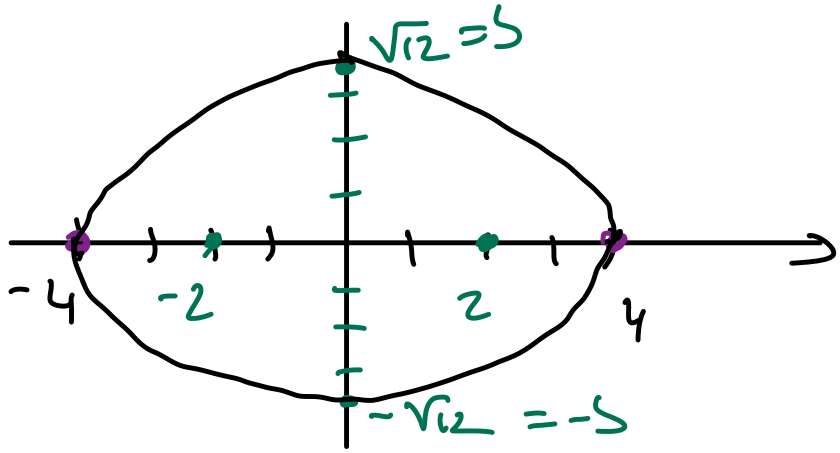
$$\boxed{\frac{x^2}{9} + \frac{y^2}{25} = 1} \qquad \boxed{\frac{x^2}{3^2} + \frac{y^2}{5^2} = 1}$$

$a^2 - b^2 = c^2$
$25 - 9 = c^2$
$4 = c$

This is an ellipse elongated vertically (along the y axis) since $5 > 3$

$a = 5$
$b = 3$

Example 3: The vertices of an ellipse are $(\pm 4, 0)$, and the foci are $(\pm 2, 0)$. Find its equation, and sketch the graph. $\hookrightarrow a = 4$ $\quad c = 2$



$b^2 = a^2 - c^2$
$b^2 = 16 - 4$
$b^2 = 12$
$b = \sqrt{12}$
$b = 2\sqrt{3}$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \frac{x^2}{4^2} + \frac{y^2}{(\sqrt{12})^2} = 1 \qquad \boxed{\frac{x^2}{16} + \frac{y^2}{12} = 1}$$

4. Graphing and Ellipse centered at (h, k)

Replacing x by $x - h$ and y by $y - k$ shifts the graph of an ellipse h units horizontally and k units vertically.

Example 4: Given $\dfrac{(x-3)^2}{16} + \dfrac{(y+2)^2}{25} = 1$, graph the ellipse and identify the center, foci and vertices.

$$\frac{(x-3)^2}{16} + \frac{(y-(-2))^2}{25} = 1$$

Center : $(3,-2)$

• elongated vertically (along y axis)

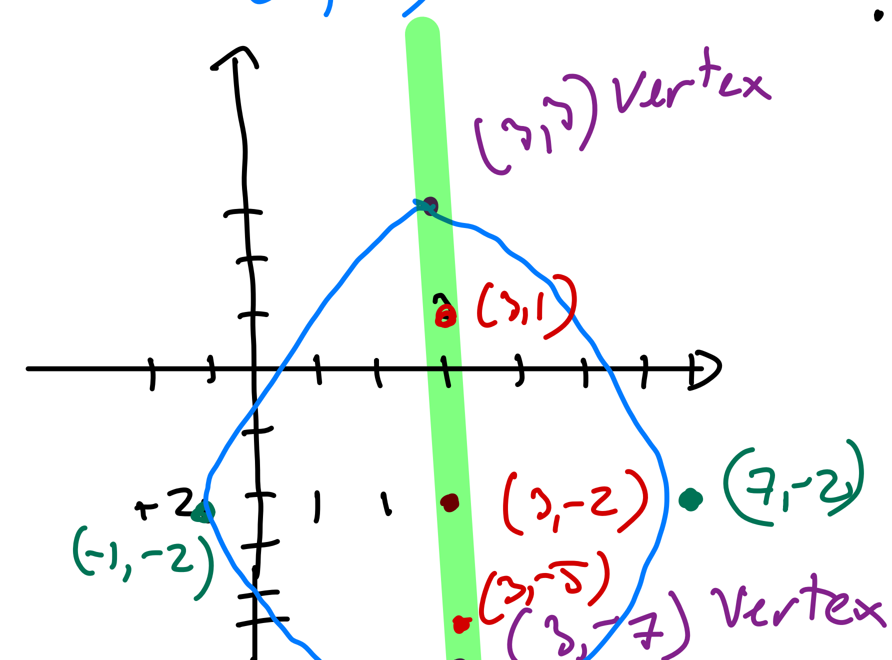
$$\frac{(x-3)^2}{4^2} + \frac{(y-(-2))^2}{5^2} = 1$$

$a = 5$; $b = 4$

$c = ?$ $\quad c^2 = a^2 - b^2$

$c^2 = 25 - 16 = 9$

$c = 3$



Example 5: Given $3x^2 + 7y^2 + 6x - 28y + 10 = 0$, write the equation of the ellipse in standard form, graph the ellipse and identify the center, foci and vertices.

Practice!