Limits Review for Improper Integrals

Calculus II

Why We Need This

Every improper integral is really a limit. For example,

$$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}\,dx \qquad \text{and} \qquad \int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx.$$

After you antidifferentiate, the whole question of *convergence* versus *divergence* comes down to evaluating a limit correctly. So the limits below need to be automatic.

1. One-Sided Limits

The notation $x \to 0^+$ means x approaches 0 **from the right**, using only values *larger* than 0 (like $0.1,\ 0.01,\ 0.001,\dots$). The notation $x \to 0^-$ means x approaches 0 **from the left**, using only values *smaller* than 0 (like $-0.1,\ -0.01,\ -0.001,\dots$).

Improper integrals with a problem at an endpoint almost always require a one-sided limit, because we only integrate on one side of the bad point.

2. The Two Limits Everyone Forgets

Key Limit 1: $\displaystyle\lim_{x\to 0^+}\frac{1}{x}$

Divide 1 by smaller and smaller positive numbers:

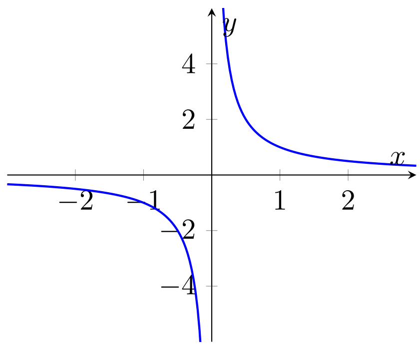
$$\frac{1}{0.1} = 10, \qquad \frac{1}{0.01} = 100, \qquad \frac{1}{0.001} = 1000, \qquad \dots$$

The outputs grow without bound. A helpful shorthand (not real arithmetic!) is

$$\frac{1}{0^+} = +\infty, \qquad \frac{1}{0^-} = -\infty.$$

Since the left and right limits differ, $\displaystyle\lim_{x\to 0}\frac{1}{x}$ does not exist.

$y = \frac{1}{x}$



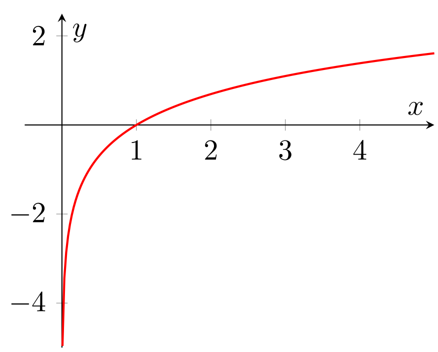
Key Limit 2: $\lim_{x \to 0^+} \ln x$

Remember that $\ln x$ answers the question "e to what power gives x?" To get a tiny positive number, you need a very negative exponent:

$$\ln(0.1) \approx -2.3, \qquad \ln(0.001) \approx -6.9, \qquad \ln(10^{-10}) \approx -23.0, \qquad \ldots$$

So $\ln x$ drops toward $-\infty$ as $x \to 0^+$. Note that $\ln x$ is *not defined* for $x \leq 0$, so only the right-hand limit makes sense.

$y = \ln x$



3. Table of Limits to Know Cold

As $x \to 0^+$	**As** $x \to \infty$
$\lim_{x\to 0^+} \frac{1}{x} = \infty$	$\lim_{x\to\infty} \frac{1}{x} = 0$
$\lim_{x\to 0^+} \frac{1}{x^p} = \infty \quad (p > 0)$	$\lim_{x\to\infty} \frac{1}{x^p} = 0 \quad (p > 0)$
$\lim_{x\to 0^+} \ln x = -\infty$	$\lim_{x\to\infty} \ln x = \infty$
$\lim_{x\to 0^+} \sqrt{x} = 0$	$\lim_{x\to\infty} \sqrt{x} = \infty$
$\lim_{x\to 0^+} e^x = 1$	$\lim_{x\to\infty} e^x = \infty$
$\lim_{x\to 0^+} e^{-x} = 1$	$\lim_{x\to\infty} e^{-x} = 0$
$\lim_{x\to 0^+} \arctan x = 0$	$\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$

Also useful: $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2}$.

4. Moving the Bad Point: Limits Near $x = a$

The same ideas work at any point. If the denominator goes to 0 while the numerator does not, the limit is $\pm\infty$; just check the **sign** of the denominator.

$$\lim_{x\to 2^+} \frac{1}{x-2} = \frac{1}{0^+} = \infty, \qquad \lim_{x\to 2^-} \frac{1}{x-2} = \frac{1}{0^-} = -\infty, \qquad \lim_{x\to 2} \frac{1}{(x-2)^2} = \frac{1}{0^+} = \infty.$$

5. Growth Rates and Indeterminate Forms

As $x \to \infty$, functions grow at very different speeds:

$$\ln x \ \ll \ x^p \ (p > 0) \ \ll \ e^x.$$

"Faster wins." For example, $\frac{\ln x}{x} \to 0$ and $\frac{x^{10}}{e^x} \to 0$ as $x \to \infty$.

Forms like $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, and $\infty-\infty$ are **indeterminate**: you cannot tell the answer just by looking. Use algebra or L'Hôpital's Rule. For $0\cdot\infty$, rewrite as a fraction first.

Warning: Forms like $\frac{1}{0^+}$, $\frac{-\infty}{0^+}$, $\infty+\infty$, and $\infty\cdot\infty$ are *not* indeterminate. Do not use L'Hôpital on them.

6. Practice Problems

Evaluate each limit. If the limit is infinite, say whether it is ∞ or $-\infty$.

1. $\displaystyle\lim_{x\to 0^+} \frac{1}{x^2}$

2. $\displaystyle\lim_{x\to 0^-} \frac{1}{x}$

3. $\displaystyle\lim_{x\to 0^+} \frac{1}{\sqrt{x}}$

4. $\displaystyle\lim_{x\to 0^+} (3 - \ln x)$

5. $\displaystyle\lim_{x\to 5^-} \frac{1}{x-5}$

6. $\displaystyle\lim_{t\to \infty} \left(1 - \frac{1}{t}\right)$

7. $\displaystyle\lim_{t\to \infty} \arctan t$

8. $\displaystyle\lim_{t\to 0^+} \left(2 - 2\sqrt{t}\right)$

9. $\displaystyle\lim_{t\to 0^+} (\ln 1 - \ln t)$

10. $\displaystyle\lim_{t\to 0^+} t\ln t$

11. $\displaystyle\lim_{t\to \infty} te^{-t}$

12. $\displaystyle\lim_{t\to 0^+} \frac{\ln t}{t}$

Connecting to Improper Integrals. Use your answers above to decide whether each integral converges or diverges.

13. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ (compare with Problem 8)

14. $\displaystyle\int_0^1 \frac{1}{x}\,dx$ (compare with Problem 9)

15. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ (compare with Problem 6)

16. $\int_0^1 \ln x \, dx$ (Hint: $\int \ln x \, dx = x \ln x - x + C$.)

7. Solutions

1. As $x \to 0^+$, $x^2 \to 0^+$, so $\dfrac{1}{x^2} \to \dfrac{1}{0^+} = \boxed{\infty}$.

2. As $x \to 0^-$, x is a tiny negative number, so $\dfrac{1}{x} \to \dfrac{1}{0^-} = \boxed{-\infty}$.

3. $\sqrt{x} \to 0^+$, so $\dfrac{1}{\sqrt{x}} \to \boxed{\infty}$.

4. $\ln x \to -\infty$, so $-\ln x \to \infty$ and $3 - \ln x \to \boxed{\infty}$.

5. For x slightly less than 5, $x-5$ is a tiny negative number, so $\dfrac{1}{x-5} \to \dfrac{1}{0^-} = \boxed{-\infty}$.

6. $\dfrac{1}{t} \to 0$, so $1 - \dfrac{1}{t} \to \boxed{1}$.

7. $\arctan t \to \boxed{\dfrac{\pi}{2}}$ (horizontal asymptote of arctan).

8. $\sqrt{t} \to 0$, so $2 - 2\sqrt{t} \to \boxed{2}$.

9. $\ln 1 = 0$ and $\ln t \to -\infty$, so $0 - \ln t \to \boxed{\infty}$.

10. This is the indeterminate form $0 \cdot (-\infty)$. Rewrite and use L'Hôpital (form $\frac{-\infty}{\infty}$):

$$\lim_{t\to 0^+} t \ln t = \lim_{t\to 0^+} \frac{\ln t}{1/t} \overset{\text{L'H}}{=} \lim_{t\to 0^+} \frac{1/t}{-1/t^2} = \lim_{t\to 0^+} (-t) = \boxed{0}.$$

11. This is $\infty \cdot 0$. Rewrite and use L'Hôpital (form $\frac{\infty}{\infty}$):

$$\lim_{t\to\infty} te^{-t} = \lim_{t\to\infty} \frac{t}{e^t} \overset{\text{L'H}}{=} \lim_{t\to\infty} \frac{1}{e^t} = \boxed{0}.$$

12. **Not** indeterminate: the numerator $\to -\infty$ and the denominator $\to 0^+$. A huge negative number divided by a tiny positive number is a huge negative number: $\dfrac{-\infty}{0^+} = \boxed{-\infty}$. (L'Hôpital does not apply here.)

13. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \Big[2\sqrt{x}\Big]_t^1 = \lim_{t\to 0^+} \left(2 - 2\sqrt{t}\right) = 2.$ **Converges** to 2.

14. $\displaystyle\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+} \Big[\ln x\Big]_t^1 = \lim_{t\to 0^+} (\ln 1 - \ln t) = \infty.$ **Diverges**.

15. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty} \left(1 - \frac{1}{t}\right) = 1.$ **Converges** to 1.

16.

$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+}\Big[\,x\ln x - x\,\Big]_t^1 = \lim_{t\to 0^+}\Big[(0-1)-(t\ln t - t)\Big] = -1-0+0 = -1.$$

(We used Problem 10: $t\ln t \to 0$.)   **Converges** to -1.