Next, we are going to introduce Geometric Sequences. A geometric sequence is generated when we start with a number a and repeatedly multiply by a fixed nonzero constant r.

Definition: A geometric sequence is a sequence of the form:

$$a, ar, ar^2, ar^3, \ldots, ar^{n-1}, \ldots$$
$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

The number a is the first ter and r is called common ratio. $\frac{a_2}{a_1} = \frac{ar}{a} = r$ or $\frac{a_4}{a_3} = \frac{ar^3}{ar^2} = r$

The nth term of a geometric sequence is given by:

$$\boxed{a_n = a r^{n-1}}$$

Example 5: If $a = 3$ and $\boxed{r = 2}$, then we have the geometric sequence:

$a_n = a r^{n-1}$ $\quad \{a_n\}: \{3, 6, 12, 24, \ldots\}$ $\quad a_{10} = ?$

$\boxed{a_n = 3 \cdot 2^{n-1}}$ $\quad a_{10} = 3 \cdot 2^{10-1} = 3 \cdot 2^9 = 1536$

Example 6: The sequence

$2, -10, 50, -250, 1250, \cdots$ is a geometric sequence with $a = 2$ and $r = -5$. Find a_{10}.

$a_n = a r^{n-1}$ $\quad a_{10} = 2(-5)^{10-1} = 2(-5)^9$

$a_n = 2(-5)^{n-1}$ $\quad a_{10} = -3{,}906{,}250$

Example 6: Find the common ratio, the first term, the nth term, and the eighth term of the geometric sequence $5, 15, 45, 135, \cdots$

$a_1 \quad a_2 \quad a_3 \quad a_4$

$r = \frac{a_2}{a_1} = \frac{15}{5} = 3$

$a = 5$ $\quad a_8 = 5(3)^{8-1}$

$a_n = a r^{n-1} \Rightarrow \boxed{a_n = 5(3)^{n-1}}$ $\quad a_8 = 5 \cdot 3^7 = 10935$

3. Partial Sums of Geometric Sequences

The partial sum s_n of a geometric sequence is given by the formula below.

PARTIAL SUMS OF A GEOMETRIC SEQUENCE

$a_1, a_2, a_3, \ldots, a_n, \ldots$

$a_n = ar^{n-1}$

For the geometric sequence defined by $\boxed{a_n = ar^{n-1}}$, the ***n*th partial sum**

$$\boxed{S_n} = \underset{a_1}{a} + \underset{a_2}{ar} + \underset{a_3}{ar^2} + \underset{a_4}{ar^3} + \underset{a_5}{ar^4} + \cdots + ar^{n-1} \qquad \boxed{r \neq 1}$$

$+ \cdots + a_n$

is given by

$$\boxed{S_n = a\frac{1 - r^n}{1 - r}} \ *$$

Example 7: Find the following partial sum of a geometric sequence:

$1 + 4 + 16 + \cdots + 4096$ $\times$

$a_1 \quad a_2 \quad a_3 \qquad a_? \quad a_7$

add the first 7 term

$$\boxed{S_7 = 1\,\frac{1-4^7}{1-4} = \frac{1-4^7}{-3} = 5461}$$

$$S_n = a\,\frac{1-r^n}{1-r}$$

$\underbrace{1, 4, 16, 64, 4096}, \ldots$

$a_n = ar^{n-1}$

$a = 1$

$r = 4$

$a_n = 1 \cdot 4^{n-1} \Rightarrow a_n = 4^{n-1}$

$4096 = 4^{n-1}$

$4^6 = 4^{n-1} \Rightarrow 6 = n-1$

$n = 7$

4. Infinite Geometric Series

An infinite geometric series is a series of the form

$a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + \cdots$ $\times$

SUM OF AN INFINITE GEOMETRIC SERIES

If $|r| < 1$, then the infinite geometric series

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \cdots$$

converges and has the sum

$$\boxed{S = \frac{a}{1 - r}} \ * \ \times$$

If $|r| \geq 1$, the series diverges. $\times$

$|r| < 1$

$\boxed{-1 < r < 1}$

Example 8: Determine whether the infinite geometric series is convergent or divergent. If it is convergent, find its sum.

(a) $2 + \frac{2}{5} + \frac{2}{25} + \frac{2}{125} + \cdots$

(b) $1 + \frac{7}{5} + \frac{49}{25} + \cdots$

a) $\frac{2}{5^0} + \frac{2}{5^1} + \frac{2}{25} + \frac{2}{125} + \cdots = \sum_{k=1}^{\infty} \frac{2}{5^k}$

$25 \to 5^2$, $125 \to 5^3$

$2 \cdot \frac{1}{5} = \frac{2}{5}$

$\frac{2}{5} \cdot \frac{1}{5} = \frac{2}{25}$

$a = 2$

$r = \frac{1}{5}$

$|r| = \left|\frac{1}{5}\right| < 1$ series converges to the value $\boxed{S = \frac{a}{1-r}} = \frac{2}{1-\frac{1}{5}} = \frac{2}{\frac{4}{5}} = 2.5$

b) $1, \frac{7}{5}, \frac{49}{25}, \cdots$ geom sequence

$1 \cdot \frac{7}{5} = \frac{7}{5}$

$\frac{7}{5} \cdot \frac{7}{5} = \frac{49}{25} \cdots$

$\boxed{a = 1 \quad r = \frac{7}{5}}$

$r = \frac{7}{5}$

$\left|\frac{7}{5}\right| \geq 1$

$\frac{7}{5} = 1.4$

so series diverges.