Mth 146: Day 17
11.9 Representations of Functions as Power Series

1 Part 1: New Concepts & Worked Examples

Representations of Functions using Geometric Series

In this section we learn how to represent some familiar functions as sums of power series. This is useful later for integrating functions that don't have elementary antiderivatives and for approximating functions by polynomials.

We start with an equation we have seen before (the geometric series with first term 1 and ratio x):

$$\frac{a}{1-r} = \frac{1}{1-x}$$

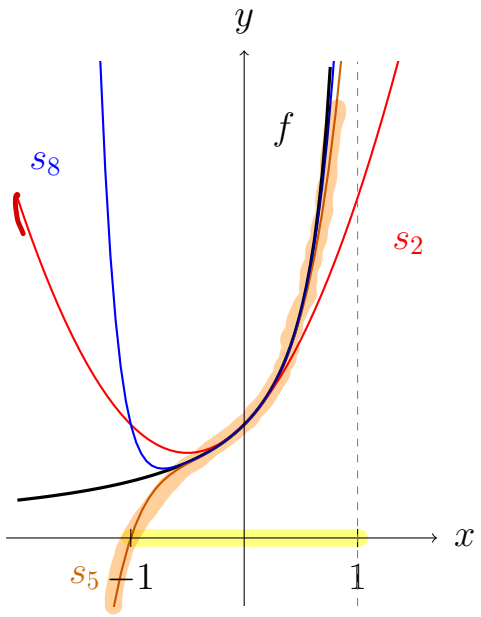
$$\textbf{(1)} \quad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

Our point of view is now different: we regard Equation (1) as expressing the *function* $f(x) = \dfrac{1}{1-x}$ as the sum of a power series. We say that $\sum_{n=0}^{\infty} x^n$, $|x| < 1$, is a power series representation of $\dfrac{1}{1-x}$ on the interval $(-1, 1)$.

Because the sum of a series is the limit of its partial sums,

$$\frac{1}{1-x} = \lim_{n\to\infty} s_n(x) \qquad \text{where} \qquad s_n(x) = 1 + x + x^2 + \cdots + x^n.$$

As n increases, $s_n(x)$ becomes a better approximation to $f(x)$ for $-1 < x < 1$.



$f(x) = \dfrac{1}{1-x}$ (black) and the partial sums s_2, s_5, s_8

$s_2 = 1 + x + x^2$

$s_5 = 1 + x + x^2 + x^3 + x^4 + x^5$

Strategy. To find a power series for a function using geometric series:

Step 1. Rewrite the function in the form $\dfrac{a}{1-r}$ (factor out constants and powers of x as needed).

Step 2. Replace x by r in Equation (1):

$$\frac{a}{1-r} = \sum_{n=0}^{\infty} ax^n.$$

Step 3. The series converges when $|x|<1$; solve this for x to get the interval of convergence. (No Ratio Test needed — it's a geometric series!)

Example 1. Express $\dfrac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence. $(-1,1)$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots$$

$$= 1 - x^2 + x^4 - x^6 + x^8 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Because this is a geometric series w/ $r = -x^2$ it converges for $|r|<1$ aka $|-x^2|<1$.

$$\frac{1}{1-*} = 1 + * + *^2 + *^3 + \cdots *^n + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

Example 2. Find a power series representation for $\dfrac{1}{x+2}$.

$$\frac{1}{x+2} = \frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2}\left(\frac{1}{1+x/2}\right)$$

$$= \frac{1}{2}\,\frac{1}{1-\left(-\frac{x}{2}\right)} = \frac{1}{2}\left(1+\left(-\frac{x}{2}\right)+\left(-\frac{x}{2}\right)^2+\left(-\frac{x}{2}\right)^3+\cdots\right)$$

$$= \frac{1}{2}\left(1-\frac{x}{2}+\left(\frac{x}{2}\right)^2-\left(\frac{x}{2}\right)^3+\left(\frac{x}{2}\right)^4-\cdots\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^n\right) = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{2^n}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{2^{n+1}}$$

$|r|<1 \Rightarrow \left|-\frac{x}{2}\right| = \frac{|x|}{2} < 1 \Rightarrow |x|<2$ $\quad (-2,2)$

Example 3. Find a power series representation for $\dfrac{x^3}{x+2}$.

$$\frac{x^3}{x+2} = x^3\left(\frac{1}{x+2}\right) = x^3\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{x^{n+1}}\right)$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+3}}{x^{n+1}}$$

Differentiation and Integration of Power Series

The sum of a power series is a function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ whose domain is the interval of convergence. We can differentiate and integrate such functions term by term, just as we would for a polynomial.

Theorem (Term-by-Term Differentiation and Integration). If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$

(ii)

$$\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in (i) and (ii) are R.

Notes:

- In (i), the sum starts at $n = 1$ because the derivative of the constant term c_0 is 0.
- In other words: the derivative (or integral) of a power series

is the sum of the derivatives (or integrals) of its terms:

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n(x-a)^n\right]$$

- **Caution:** The *radius* of convergence stays the same, but the *interval* of convergence may change at the endpoints. The original series might converge at an endpoint where the differentiated series diverges.

Example 4. Express $\dfrac{1}{(1-x)^2}$ as a power series by differentiating Equation (1). What is the radius of convergence?

$$\frac{1}{1-x} = 1+x+x^2+x^3+\cdots = \sum_{n=0}^{\infty} x^n \quad \longrightarrow \quad |x|<1, \; R=1$$

Differentiating each side:

$$\left(\frac{1}{1-x}\right)' = \left(1+x+x^2+x^3+\cdots\right)'$$

$$\frac{1}{(1-x)^2} = 0+1+2x+3x^2+4x^3+\cdots = 1+2x+3x^2+4x^3+\cdots$$

(n=1, n=2, n=3, n=4 ...)

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \quad (*)$$

Example 5. Find a power series representation for $\ln(1+x)$ and its radius of convergence.

$$\left(\ln(1+x)\right)' = \frac{1}{1+x} = \frac{1}{1-(-x)}$$

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \left(\frac{1}{1-(-x)}\right) dx$$

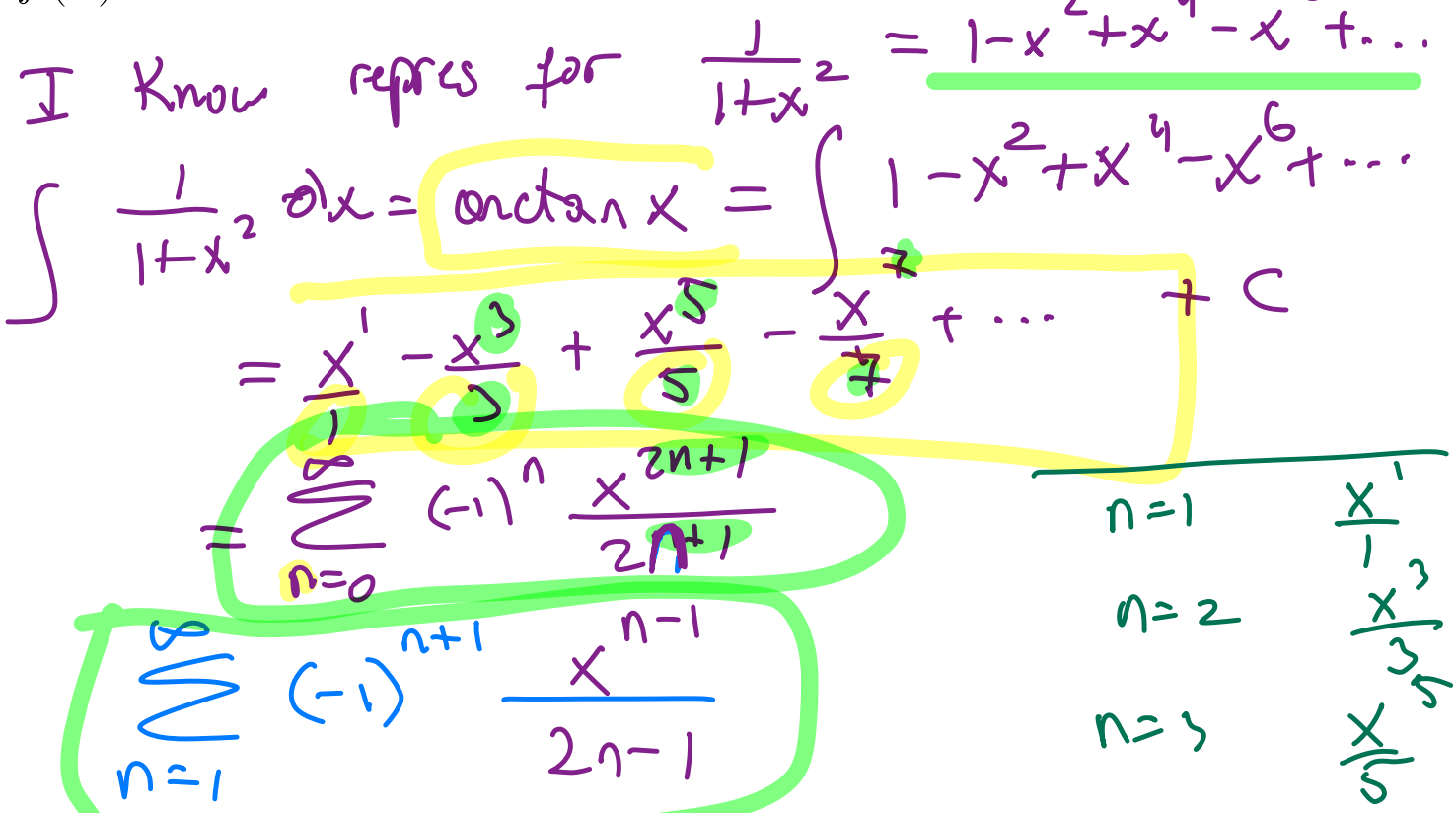
$$= \int \left(1+(-x)+(-x)^2+(-x)^3+\cdots\right) dx$$

$$= \int \left(1-x+x^2-x^3+x^4-\cdots\right) dx$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots + C = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} + C$$

To find C we let $x=0 \Rightarrow \ln(1+0) = C \Rightarrow C=0$

Example 6. Find a power series representation for $f(x) = \tan^{-1} x$.



Fun fact: This series is called **Gregory's series**. It turns out to be valid at $x = \pm 1$ too, and plugging in $x = 1$ gives the **Leibniz formula for** π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

2 Part 2: Now You Try It

Find a power series representation for each function and determine the **radius of convergence** (and the interval of convergence when asked). Final answers are given as footnotes at the bottom of the page — only the final answer, not the steps, so use them to check your work rather than replace it.

1. $f(x) = \dfrac{2}{3-x}$ (also find the interval of convergence)

[1]

2. $f(x) = \dfrac{x}{9+x^2}$ (also find the interval of convergence)

[2]

[1] Answer: $\displaystyle\sum_{n=0}^{\infty} \frac{2x^n}{3^{n+1}}$; $R = 3$; interval $(-3, 3)$.

[2] Answer: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{9^{n+1}}$; $R = 3$; interval $(-3, 3)$.

3. $f(x) = \dfrac{1}{1+4x^2}$ (also find the interval of convergence)

[3]

4. $f(x) = \dfrac{1}{(1+x)^2}$ (Hint: differentiate the series for $\frac{1}{1+x}$.)

[4]

[3] Answer: $\sum_{n=0}^{\infty}(-1)^n 4^n x^{2n}$; $R = \frac{1}{2}$; interval $(-\frac{1}{2}, \frac{1}{2})$.

[4] Answer: $\sum_{n=0}^{\infty}(-1)^n (n+1)x^n$; $R = 1$.

5. $f(x) = \ln(5 - x)$ (Hint: integrate the series for $\frac{-1}{5-x}$.)
[5]

6. $f(x) = x^2 \tan^{-1}(x^3)$ (Hint: use the series for $\tan^{-1} x$ from Example 6.)
[6]

[5] Answer: $\ln 5 - \sum_{n=1}^{\infty} \frac{x^n}{n\,5^n}$; $R = 5$.

[6] Answer: $\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+5}}{2n+1}$; $R = 1$.

7. Evaluate the indefinite integral $\displaystyle\int \frac{t}{1-t^8}\,dt$ as a power series. What is the radius of convergence?

7

[7] Answer: $C + \displaystyle\sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}$; $R = 1$.

3 Part 3: Reflection

For each item below, rate your comfort level and jot a quick note: What clicked? What's still fuzzy? What would help?

1. **Rewrite a function in the form $\dfrac{a}{1-r}$ and use the geometric series to find its power series.** 1 2 3 4 5

2. **Find the interval of convergence of a geometric-type series by solving $|r| < 1$.** 1 2 3 4 5

3. **Multiply a known series by a power of x (or substitute x^k, $-x$, $\frac{x}{2}$, ...) to get a new series.** 1 2 3 4 5

4. **Differentiate or integrate a power series term by term (and find the constant C).** 1 2 3 4 5

5. **Decide whether to substitute, multiply, differentiate, or integrate to get a series from one I already know.** 1 2 3 4 5