Mth 146: Day 16
11.8 Power Series

1 Part 1: New Concepts & Worked Examples

Power Series

So far we have studied series of numbers: $\sum a_n$. Now we consider series in which each term includes a power of the variable x.

Definition. A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0x^0 + c_1x^1 + c_2x^2 + c_3x^3 + \ldots + c_nx^n + \cdots$$
$$= c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots + c_nx^n + \cdots$$

where x is a variable and the c_n's are constants called the coefficients of the series.

For each number we substitute for x, the power series is a series of constants that we can test for convergence or divergence. A power series may converge for some values of x and diverge for others. The sum of the series is a function (s before it was a constant)

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots$$

whose domain is the set of all x for which the series conv. Notice that f resembles a polynomial — the only difference is that f has infinitely many terms.

Example: the geometric series. If we take $c_n = 1$ for all n, the power series becomes

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

which converges when $|x| < 1$ and diverges when $|x| \geq 1$.

- At $x = \frac{1}{2}$: $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n =$ converges.
- At $x = 2$: $\sum_{n=0}^{\infty} 2^n =$ diverges.

Definition. More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

is called a **power series in** $(x-a)$, or a power series centered at a (or a **power series about** a).

Notes:

- For the $n = 0$ term we use the convention $(x-a)^0 = 1$, even when $x = a$.
- When $x = a$, all of the terms are 0 for $n \geq 1$, so a power series always converges at its center, $x = a$.

Key idea: To find the values of x for which a power series converges, we normally use the Ratio Test (or the Root Test).

Example 1. For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

a_n

Apply The Ratio Test with $a_n = \frac{(x-3)^n}{n}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(x-3)^{n+1}}{n+1}}{\frac{(x-3)^n}{n}} \right| =$$

$$= \lim_{n\to\infty} \left| \frac{(x-3)^n (x-3)}{n+1} \cdot \frac{n}{(x-3)^n} \right| =$$

$L<1$ conv
$L>1$ div
$L=1$ incon.

$$= \lim_{n\to\infty} \left| \frac{n}{n+1} (x-3) \right| = \lim_{n\to\infty} \frac{n}{n+1} |x-3| =$$

$$= \lim_{n\to\infty} \frac{n/n}{\frac{n}{n}+\frac{1}{n}} |x-3| = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} |x-3| = |x-3|$$

$\frac{1}{n} \to 0$

By Ratio Test, the series is absolutely conv (and therefore convergent) when the limit is <1

So we solve $|x-3| < 1$ to find the x's it converges for.

$|x-3|<1 \iff -1 < x-3 < 1 \quad |+3$

$2 < x < 4$

So the power series converges for $2 < x < 4$.

diverges for $|x-3|>1 \Rightarrow x>4$ or $x<2$

The Ratio Test is inconclusive when limit is $= 1$ aka when

$|x-3| = 1 \Rightarrow$ $x = 2$, $x = 4$

$\Downarrow$

$x - 3 = 1$ $\quad$ $x - 3 = -1$

$x = 4$ $\quad$ $x = 2$.

At $x = 2$ $\quad \sum_{n=1}^{\infty} \frac{(x-3)^n}{n} = \sum_{n=1}^{\infty} \frac{(2-3)^n}{n} =$

$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by Alternating Series Test.

At $x = 4$ $\quad \sum_{n=1}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series).

So the power series converges for

$$2 \leq x < 4$$

Example 2. For what values of x is the series $\sum_{n=0}^{\infty} n!\, x^n$ (a_n) convergent?

Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!\, x^{n+1}}{n!\, x^n}\right| = \lim_{n\to\infty} \left|\frac{n!(n+1)x^n \cdot x}{n!\, x^n}\right|$

$= \lim_{n\to\infty} (n+1)|x| = \infty$

By RT the power series diverges for all x except for $x=0$.

It converges only when $x=0$.

$(2(n+1))! =$

Example 3. For what values of x does the series $\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$ (a_n) converge?

RT: $\lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{(2(n+1))!}}{\frac{x^n}{(2n)!}}\right| = \lim_{n\to\infty} \left|\frac{x^n \cdot x}{(2n)!(2n+1)(2n+2)} \cdot \frac{(2n)!}{x^n}\right|$

$= \lim_{n\to\infty} \left|\frac{x}{(2n+1)(2n+2)}\right| = \lim_{n\to\infty} \frac{1}{(2n+1)(2n+2)}|x| = 0 < 1$

for all x so by RT this power series converges for all x.

Radius and Interval of Convergence

For every power series we have looked at so far, the set of x-values where the series converges has turned out to be an interval. This is true in general.

Theorem. For a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

(i) The series converges only for $x=a$ (@ its center) (ex 2)

(ii) The series converges for all x (ex 3)

(iii) There is a positive number R such that the series converges for $|x-a|<R$ and diverges if $|x-a|>R$ (ex 1)

$|x-3|<1$ $\quad$ $|x-3|>1$

Definitions.

The number R in case (iii) is called the radius of convergence of the power series. By convention, $R = 0$ in case (i) and $R = \infty$ in case (ii).

The interval of convergence of a power series is the interval that consists of all values of x for which the power series converges.

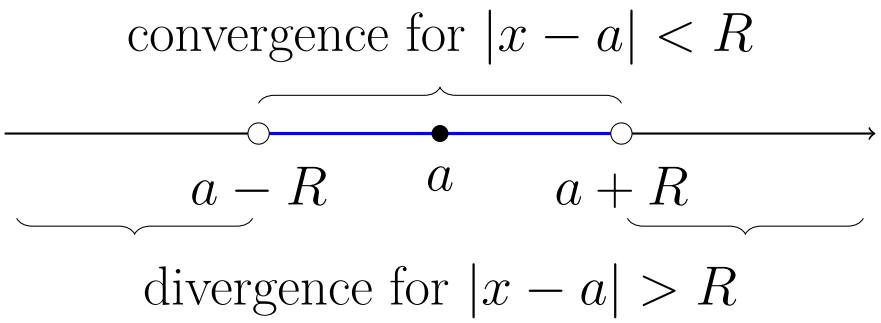
- In case (i), the interval is the single point $\{a\}$.
- In case (ii), the interval is $(-\infty, \infty)$
- In case (iii), $|x-a|<R$ can be rewritten as $(a-R, a+R)$

$|x-3|<1$

The endpoints. When x is an *endpoint*, $x = a \pm R$, anything can happen — the series might converge at one or both endpoints, or diverge at both. So in case (iii) there are four possibilities for the interval of convergence:

$(2,4)$, $[2,4)$, $(2,4]$

$[2,4]$



Summary of the examples so far:

	Series	**Radius**	**Interval**
Geometric	$\sum_{n=0}^{\infty} x^n$	1	$(-1,1)$
Example 1	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	1	$[2,4)$
Example 2	$\sum_{n=0}^{\infty} n!\, x^n$	0	$\{0\}$
Example 3	$\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$	∞	$(-\infty,\infty)$

Note: Use the Ratio Test (or sometimes the Root Test) to find the radius of convergence R. The Ratio and Root Tests **always fail** when x is an endpoint of the interval of convergence, so the endpoints must be checked with other tests.

Step 3 $|x| < 1/3$

$-1/3 < x < 1/3$

Endpoints: to check $x = -1/3$: $\sum_{n=0}^{\infty} \frac{(-3)^n (-1/3)^n}{\sqrt{n+1}} =$

$= \sum_{n=0}^{\infty} \frac{(-3)^n \frac{(1)^n}{(-3)^n}}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$

diverges (p-series w/ $p = 1/2$)

$x = 1/3$ $\sum_{n=0}^{\infty} \frac{(-3)^n (1/3)^n}{\sqrt{n+1}} =$

$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^n \frac{1}{3^n}}{\sqrt{n+1}}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges by AST

Step 4 IOC :

$\left(-1/3, 1/3\right]$

Example 5. Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}.$$

Solution: If $a_n = \dfrac{n(x+2)^n}{3^{n+1}}$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n}\right|$$
$$= \left(1+\frac{1}{n}\right)\frac{|x+2|}{3} \;\to\; \frac{|x+2|}{3} \quad \text{as } n \to \infty.$$

By the Ratio Test, the series converges if $\dfrac{|x+2|}{3} < 1$ and diverges if $\dfrac{|x+2|}{3} > 1$. So it converges if $|x+2| < 3$ and diverges if $|x+2| > 3$: the radius of convergence is $\mathbf{R = 3}$.

The inequality $|x+2| < 3$ can be written as $-5 < x < 1$, so test the endpoints -5 and 1:

- $x = -5$: $\displaystyle\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n n$, which **diverges** by the Test for Divergence ($(-1)^n n$ doesn't converge to 0).
- $x = 1$: $\displaystyle\sum_{n=0}^{\infty} \frac{n\,3^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty} n$, which also **diverges** by the Test for Divergence.

Thus the series converges only when $-5 < x < 1$, so the interval of convergence is $(\mathbf{-5, 1})$.

Find the **radius of convergence** and **interval of convergence** of each series.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^n n\,x^n$

   Solution: $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{(n+1)|x|^{n+1}}{n|x|^n} = \left(1+\dfrac{1}{n}\right)|x| \to |x|$.
   Converges if $|x|<1$, so $R=1$.
   $x=1$: $\sum(-1)^n n$ diverges (Test for Divergence). $x=-1$: $\sum(-1)^n n(-1)^n = \sum n$ diverges (Test for Divergence).
   $\mathbf{R=1}$, interval $\mathbf{(-1,1)}$.

2. $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{2n-1}$

   Solution: $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{2n-1}{2n+1}|x| = \dfrac{2-\frac{1}{n}}{2+\frac{1}{n}}|x| \to |x|$, so $R=1$.
   $x=1$: $\displaystyle\sum\frac{1}{2n-1}$ diverges (Limit Comparison with $\sum\frac{1}{n}$: $\frac{1/(2n-1)}{1/n} = \frac{n}{2n-1} \to \frac{1}{2} > 0$). $x=-1$: $\displaystyle\sum\frac{(-1)^n}{2n-1}$ converges by the AST.
   $\mathbf{R=1}$, interval $\mathbf{[-1,1)}$.

3. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Solution: $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{|x|^{n+1}}{(n+1)!} \cdot \dfrac{n!}{|x|^n} = \dfrac{|x|}{n+1} \to 0 < 1$ for every x.

$\mathbf{R} = \infty$, interval $(-\infty, \infty)$.

4. $\displaystyle\sum_{n=1}^{\infty} n^n x^n$

Solution: Root Test: $\sqrt[n]{|n^n x^n|} = n|x| \to \infty$ for every $x \neq 0$, so the series diverges unless $x = 0$.

$\mathbf{R = 0}$, interval $\{\mathbf{0}\}$.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2 x^n}{2^n}$

Solution:

$\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{(n+1)^2|x|^{n+1}}{2^{n+1}} \cdot \dfrac{2^n}{n^2|x|^n} = \left(1+\dfrac{1}{n}\right)^2 \dfrac{|x|}{2} \to \dfrac{|x|}{2}.$

Converges if $|x| < 2$, so $R = 2$.

$x = 2$: $\sum(-1)^n n^2$ diverges. $\quad x = -2$: $\sum n^2$ diverges. (Both by the Test for Divergence.)

R = 2, interval $(\mathbf{-2}, \mathbf{2})$.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1}$

Solution: $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{n^2+1}{(n+1)^2+1}|x-2| \to |x-2|.$

Converges if $|x-2| < 1$, i.e. $1 < x < 3$, so $R = 1$.

$x = 3$: $\displaystyle\sum \frac{1}{n^2+1}$ converges (Comparison with $\sum \frac{1}{n^2}$).

$x = 1$: $\displaystyle\sum \frac{(-1)^n}{n^2+1}$ converges (absolutely, by the same comparison).

R = 1, interval $[\mathbf{1}, \mathbf{3}]$.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$

Solution: $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{|2x-1|^{n+1}}{5^{n+1}\sqrt{n+1}} \cdot \dfrac{5^n\sqrt{n}}{|2x-1|^n} =$
$\sqrt{\dfrac{n}{n+1}}\,\dfrac{|2x-1|}{5} \to \dfrac{|2x-1|}{5}.$
Converges if
$|2x-1| < 5 \iff \left|x - \frac{1}{2}\right| < \frac{5}{2} \iff -2 < x < 3$, so
$R = \frac{5}{2}$.
$x = 3$: $\sum \dfrac{5^n}{5^n\sqrt{n}} = \sum \dfrac{1}{\sqrt{n}}$ diverges ($p = \frac{1}{2} \le 1$).
$x = -2$: $\sum \dfrac{(-5)^n}{5^n\sqrt{n}} = \sum \dfrac{(-1)^n}{\sqrt{n}}$ converges by the AST.
$\mathbf{R = \frac{5}{2}}$, interval $[\mathbf{-2}, \mathbf{3})$.

3 Part 3: Reflection

For each item below, rate your comfort level and jot a quick note: What clicked? What's still fuzzy? What would help?

1. **Recognize a power series $\sum c_n(x-a)^n$ and identify its center a and coefficients c_n.** 1 2 3 4 5

2. **Use the Ratio Test to find the values of x where a power series converges.** 1 2 3 4 5

3. **Find the radius of convergence R, including the cases $R = 0$ and $R = \infty$.** 1 2 3 4 5

4. **Test the endpoints with another test (Test for Divergence, p-series, Comparison, AST).** 1 2 3 4 5

5. **Write the interval of convergence with the correct brackets at each endpoint.** 1 2 3 4 5