Mth 146: Day 20
11.11 Approximating Functions by Taylor Polynomials

1 Part 1: New Concepts & Worked Examples

Taylor Polynomials as Approximations

Suppose f is equal to the sum of its Taylor series at a. In *Taylor and Maclaurin Series* we called the nth partial sum of that series the **nth-degree Taylor polynomial of f at a**:

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$= \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$\underline{\qquad\qquad\qquad\qquad\qquad}$$

Since $T_n(x) \to f(x)$ as $n \to \infty$, we can use T_n as an approximation:

$$f(x) \approx \underline{\qquad\qquad}$$

Key facts:

- The first-degree Taylor polynomial $T_1(x) = \underline{\qquad\qquad\qquad\qquad}$ is the same as the $\underline{\qquad\qquad\qquad\qquad}$ of f at a (the tangent line).
- **T_n and f have the same derivatives at a** up to and including order $\underline{\qquad}$.

- **The farther x is from a, the _____ $T_n(x)$ converges to $f(x)$**, so we need a higher degree to get the same accuracy.

For $f(x) = e^x$ at $a = 0$, compare the Taylor polynomials with the true value:

	$x = 0.2$	$x = 3.0$
$T_2(x)$	1.220000	8.500000
$T_4(x)$	1.221400	16.375000
$T_6(x)$	1.221403	19.412500
$T_8(x)$	1.221403	20.009152
$T_{10}(x)$	1.221403	20.079665
e^x	1.221403	20.085537

At $x = 0.2$ (close to 0) the convergence is _____; at $x = 3$ (far from 0) it is _____.

Important notes:

How Good Is the Approximation?

When we use T_n to approximate f, we want to know: **How big is the error? How large must n be to get a desired accuracy?** The error is the absolute value of the remainder:

$$|R_n(x)| = _____$$

Three ways to estimate the error:

1. Use a calculator or computer to _____ $|R_n(x)|$.
2. If the series is _____, use the **Alternating Series Estimation Theorem**.
3. In **all** cases, we can use _____.

Recall the two error tools:

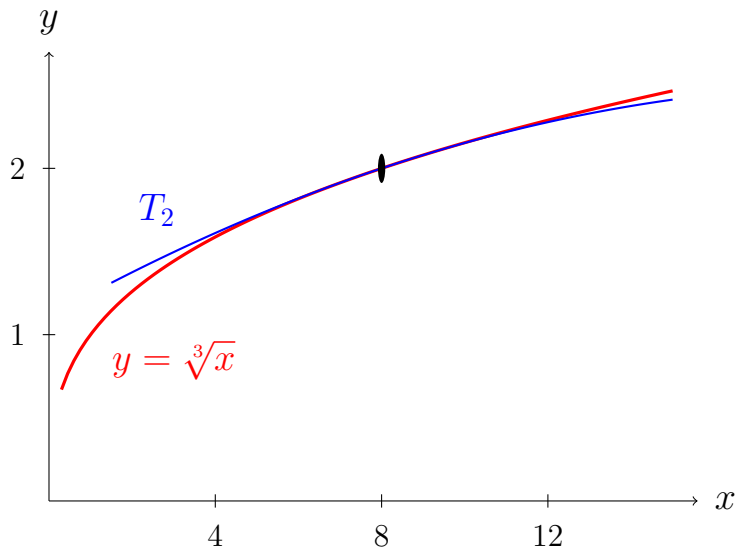
Taylor's Inequality. If $\left|f^{(n+1)}(x)\right| \leq M$ for $|x-a| \leq d$, then

$$|R_n(x)| \leq _____ \qquad \text{for } |x-a| \leq d$$

Alternating Series Estimation Theorem. If $\sum(-1)^{n-1}b_n$ satisfies $b_{n+1} \leq b_n$ and $b_n \to 0$, then the error from stopping early is at most

$$|R_n| \leq ____ \qquad \textbf{(the first term we left out)}$$

Example 1. (a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.
(b) How accurate is this approximation when $7 \leq x \leq 9$?



$y = \sqrt[3]{x}$ (red) and $y = T_2(x)$ (blue) are very close near $x = 8$

Checking with a graph: graphing $|R_2(x)| = |\sqrt[3]{x} - T_2(x)|$ on $7 \leq x \leq 9$ shows $|R_2(x)| < 0.0003$. So the graph gives a _____ estimate than Taylor's Inequality.

Taylor's Inequality gives an upper bound, not the exact error.

Important notes:

Summary: Which error method should I use?

- Series is alternating (with decreasing terms) $\Rightarrow$ _____.
- Series is not alternating (or you are not sure) $\Rightarrow$ _____.
- Want to check or improve the estimate $\Rightarrow$ _____.

Why this matters: when you press the sin or e^x key on a calculator, the machine often computes a _____, usually a modified Taylor polynomial.

Important notes: