Day 14

- Piece wise practice counts as written HW 3
- Exam reflection counts as written HW 4

1. Adding, subtracting, multiplying and dividing functions

Two functions f and g can be combined to form new functions $f+g$, $f-g$, fg, and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers.

ALGEBRA OF FUNCTIONS

Let f and g be functions with domains A and B. Then the functions $f+g$, $f-g$, fg, and f/g are defined as follows.

$(f+g)(x) = f(x) + g(x)$	Domain $A \cap B$
$(f-g)(x) = f(x) - g(x)$	Domain $A \cap B$
$(fg)(x) = f(x)g(x)$	Domain $A \cap B$
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	Domain $\{x \in A \cap B \mid g(x) \neq 0\}$

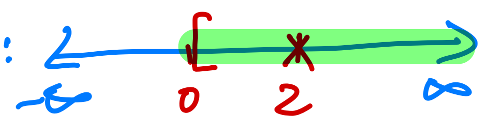
Practice: Perform the following function operations for f and g and find the domains of the resulting functions.

$f(x) = \frac{1}{x-2}$ and $g(x) = \sqrt{x}$.

Domain f: $x \neq 2$ $(-\infty, 2) \cup (2, \infty)$
g: $x \geq 0$ $[0, \infty)$

(a) $(f+g)(x) = \frac{1}{x-2} + \sqrt{x}$

Domain $f+g$:



$[0,2) \cup (2,\infty)$

(b) $(f-g)(x) = \frac{1}{x-2} - \sqrt{x}$

Domain $f-g$: $[0,2) \cup (2,\infty)$

(c) $(fg)(x) = \frac{1}{x-2} \cdot \sqrt{x} = \frac{\sqrt{x}}{x-2}$ Domain: fg $(0,2) \cup (2,\infty)$

(d) $\left(\frac{f}{g}\right)(x) = \frac{\frac{1}{x-2}}{\sqrt{x}} = \frac{1}{x-2} \cdot \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{x}(x-2)}$

Domain $\frac{f}{g}$: $(0,2) \cup (2,\infty)$

2. Composition of Functions

Another way to combine two functions f and g is to compose them. Composing the function $f(x)$ with the function $g(x)$, denoted $(f \circ g)(x)$, means that the output of g becomes the input of f. In other words $(f \circ g)(x) = f(g(x))$ or the function f is evaluated at $g(x)$.

For example if $f(x) = 5x - 3$ and $g(x) = 2 - x^2$ then $(f \circ g)(x) = f(g(x))$

D: $(-\infty, \infty)$ (for f); D: $(-\infty, \infty)$ (for g)

$$= 5g(x) - 3$$
$$= 5(2 - x^2) - 3$$
$$= 10 - 5x^2 - 3 = 7 - 5x^2$$

$f \circ g \neq g \circ f$

What is the domain of $f \circ g$?

D: $(-\infty, \infty)$

$$(g \circ f)(x) = g(f(x)) = 2 - (f(x))^2 = 2 - (5x-3)^2$$
$$= 2 - (5x-3)(5x-3)$$
$$= 2 - (25x^2 - 15x - 15x + 9) = -25x^2 + 30x - 7$$

D: $(-\infty, -2) \cup (-2, \infty)$

Example: Let $f(x) = \dfrac{1}{x+2}$ and $g(x) = \dfrac{4}{x-1}$. Find the following:

D: $(-\infty, 1) \cup (1, \infty)$

(a) $f \circ g$ and its domain

$$(f \circ g)(x) = f(g(x)) = \frac{1}{g(x)+2} = \frac{1}{\frac{4}{x-1} + \frac{2}{1}\cdot\frac{(x-1)}{(x-1)}} = \frac{1}{\frac{4+2(x-1)}{x-1}}$$

$$= 1 \cdot \frac{x-1}{4+2x-2} = \frac{x-1}{2+2x}$$

$$(f \circ g)(x) = \frac{x-1}{2+2x} = \frac{x-1}{2(x+1)}$$

D: $f \circ g$: $-\infty$, -1, 1, ∞ $\quad (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

(b) $g \circ f$ and its domain

$$(g \circ f)(x) = g(f(x)) = \frac{4}{f(x) - 1} = \frac{4}{\frac{1}{x+2} - \frac{1}{1}\cdot\frac{(x+2)}{(x+2)}} =$$

$$= \frac{4}{\frac{1-(x+2)}{x+2}} = \frac{4}{\frac{1-x-2}{x+2}} = \frac{4}{\frac{-x-1}{x+2}}$$

$$= 4 \cdot \frac{x+2}{-x-1} = \frac{4(x+2)}{-(x+1)} = g \circ f$$

D: $g \circ f$: $-\infty$, -2, -1, ∞

$(-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$

NOTE The domain of $f \circ g$ is the intersection of the domain of inner function g and the resulting function $f \circ g$.

3. Applications with composition of functions

The weekly cost C of producing x units is given by $C(x) = 60x + 750$. The number x of units produced in t hours is given by $x(t) = 50t$.

$50(4.75) = 238$ units

(a) Find and interpret $(C \circ x)(t)$ $= C(x(t)) = 60\,x(t) + 750$

$= 60(50t) + 750 = 3000t + 750$

$(C \circ x)(t)$ is the function that gives you weekly cost in terms of hours worked in a week.

(b) Find the time that must elapse in order for the cost to increase to $15,000.

I want the number of hours worked when cost is 15,000. Use $(C \circ x)(t)$ function

$(C \circ x)(t) = 3000t + 750$

$15000 = 3000t + 750$ $\Rightarrow 14250 = 3000t$ $\mid \div 3000$

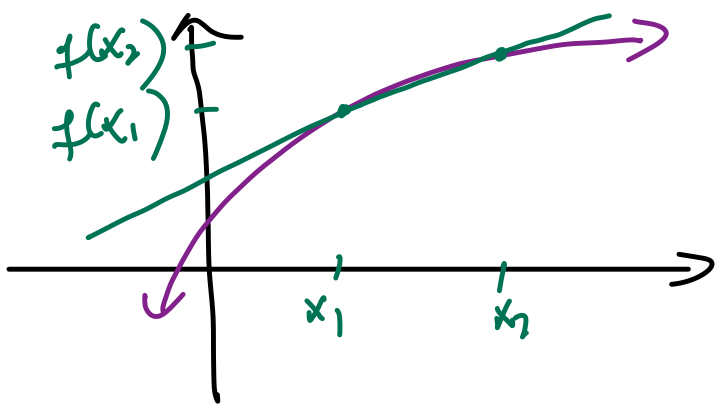
$t = ?$

hours $4.75 = t$

4. Difference Quotient

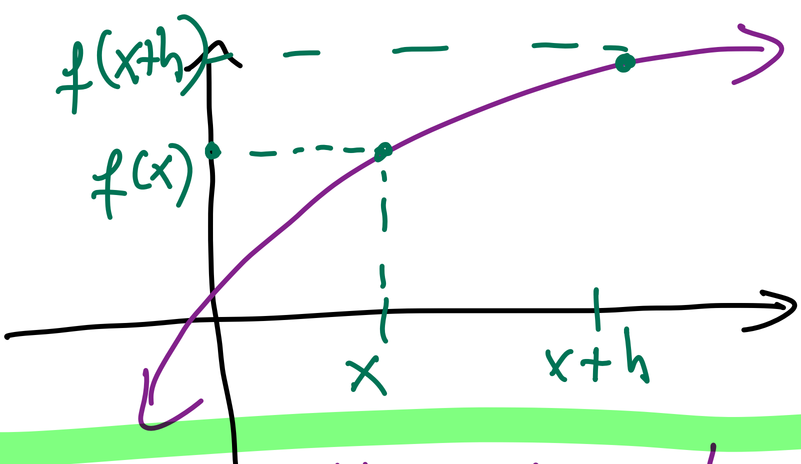
The difference quotient represents the average rate of change of a function between two values x and $x + h$.

Recall that the Average rate of change of a function f between two points x_1 and x_2 is just the slope of the line going through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.



$$ARC = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Formula:



$$\frac{f(x+h) - f(x)}{\cancel{x} + h - \cancel{x}}$$

Difference quotient formula: $\dfrac{f(x+h) - f(x)}{h}$

$(x+h)^2 = (x+h)(x+h) = x^2 + xh + hx + h^2$

Example: Find the difference quotient for $\boxed{f(x) = 2x^2 + 3x + 1}$.

$$\frac{f(x+h)-f(x)}{h} = \frac{2(x+h)^2+3(x+h)+1-(2x^2+3x+1)}{h}$$

$$= \frac{2(x^2+2xh+h^2)+3x+3h+1-2x^2-3x-1}{h}$$

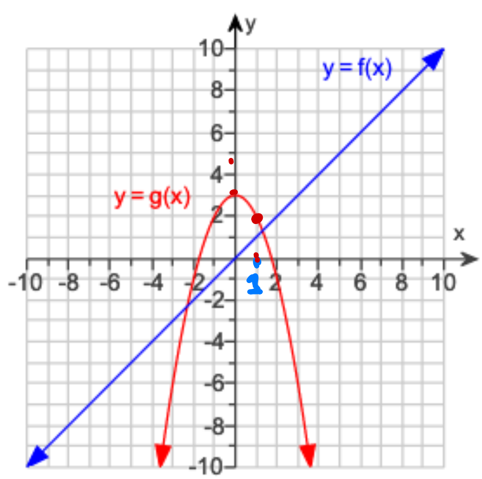
$$= \frac{\cancel{2x^2}+4xh+2h^2+\cancel{3x}+3h\cancel{+1}\cancel{-2x^2}\cancel{-3x}\cancel{-1}}{h}$$

$$= \frac{4xh+2h^2+3h}{h} = \frac{\cancel{h}(4x+2h+3)}{\cancel{h}} = 4x+2h+3$$

(Goal when calculating difference quotient is to cancel the h at the denominator)

Practice: Exercise 1

Use a graphical approach to answer the following questions about the two functions graphed. Find $(f+g)(0), (f-g)(-1), (fg)(1), (f/g)(2), (f \circ g)(1), (g \circ f)(1)$.



$\left(\frac{f}{g}\right)(2) = \frac{f(2)}{g(2)} = \frac{2}{-1} = -2$

$(f \circ g)(1) = f(g(1))$
$= f(2) = 2$

$(g \circ f)(1) = g(f(1))$
$= g(1)$
$= 2$

$(f+g)(0) = f(0) + g(0) = 0 + 3 = 3$

$(f-g)(-1) = f(-1) \cancel{+} g(-1) = -1 \cancel{+} 2 = 3$

$(fg)(1) = f(1)\,g(1) = (1)(2) = 2$

Practice: Exercise 2

Let $f(x) = \dfrac{x}{x+3}$ and $g(x) = 8x - 3$. Find the following

1. $(f \circ g)(x)$ and its domain

$$f(g(x)) = \frac{g(x)}{g(x)+3} = \frac{8x-3}{8x-3+3} = \frac{8x-3}{8x}$$

Domain: $x \neq 0$ $\quad (-\infty, 0) \cup (0, \infty)$

(g(x) is not giving us any problems)

2. $(g \circ f)(x)$ and its domain

$$g(f(x)) = 8f(x) - 3 = 8\frac{x}{x+3} - 3 = \frac{8x}{x+3} - \frac{3}{1}\frac{(x+3)}{(x+3)}$$

$$= \frac{8x - 3(x+3)}{x+3} = \frac{8x-3x-9}{x+3} = \frac{5x-9}{x+3}$$

Domain: $x \neq -3$ $\quad (-\infty, -3) \cup (-3, \infty)$

Practice: Exercise 3

Let $f(x) = x^3 + 8$ and $g(x) = x - 5$ and $h(x) = \sqrt{x}$. Find $f \circ g \circ h$.

$$(f \circ g \circ h)(x) = f(g(h(x)))$$
$$= f(h(x) - 5)$$
$$= f(\sqrt{x} - 5)$$
$$= (\sqrt{x} - 5)^3 + 8$$

$$V_{sphere} = \frac{4}{3}\pi r^3$$

Practice: Exercise 4

A spherical balloon is being inflated. The radius of the balloon is increasing at a rate of 3cm/s.

1. Find a function f that models the radius as a function of time t, in seconds.

$$f(t) = 3t$$

2. Find a function g that models the volume as a function of the radius r, in cm.

$$\boxed{g(r) = \frac{4}{3}\pi r^3}$$

3. Find and interpret $g \circ f$.

$$(g \circ f)(t) = g(f(t)) = \frac{4}{3}\pi (f(t))^3$$

$$= \frac{4}{3}\pi (3t)^3 = \frac{4}{\cancel{3}}\pi \cancel{27}^{\,9} t^3$$

related

$$V(t) = 36\pi t^3$$

volume

Practice: Exercise 5

Find the difference quotient for $f(x) = \dfrac{1}{x+1}$.

$$\frac{f(x+h) - f(x)}{h} = \frac{\dfrac{1}{(x+h+1)}\dfrac{(x+1)}{(x+1)} - \dfrac{1}{(x+1)} \cdot \dfrac{(x+h+1)}{(x+h+1)}}{h}$$

$$= \frac{\dfrac{\cancel{x}+\cancel{1} - \cancel{x} - h - \cancel{1}}{(x+h+1)(x+1)}}{h} = \frac{\dfrac{-h}{(x+h+1)(x+1)}}{h}$$

$$= -\frac{\cancel{h}}{(x+h+1)(x+1)} \cdot \frac{1}{\cancel{h}} = -\frac{1}{(x+h+1)(x+1)}$$