Example 3. $\int \sin^5(x)\cos^2(x)\,dx$

odd power of sine
$u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$$= \int \sin^4 x \cos^2(x) \sin x\,dx$$

$$\sin^4 x = (\sin^2 x)^2 = (1-\cos^2 x)^2 = (1-u^2)^2$$

$$= -\int (1-u^2)^2 u^2\,du$$

$$= -\int (1-2u^2+u^4)u^2\,du$$

$(a+b)^2 = a^2+2ab+b^2$

$$= -\int u^2 - 2u^4 + u^6\,du$$

$$= -\int u^2\,du + 2\int u^4\,du - \int u^6\,du$$

$$= -\frac{u^3}{3} + 2\frac{u^5}{5} - \frac{u^7}{7} + C = \boxed{-\frac{\cos^3 x}{3} + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C}$$

Example 4. $\int \sin^4(x)\cos^2(x)\,dx$

$$= \int (\sin^2(x))^2 \cos^2(x)\,dx$$

$$= \int \left[\frac{1}{2}(1-\cos(2x))\right]^2 \left(\frac{1}{2}\right)(1+\cos(2x))\,dx$$

$$= \frac{1}{4}\cdot\frac{1}{2}\int (1-2\cos(2x)+\cos^2(2x))(1+\cos(2x))\,dx$$

$$= \frac{1}{8}\int 1+\cos(2x)-2\cos(2x)-2\cos^2(2x) + \cos^2(2x)+\cos^3(2x)\,dx$$

$$= \frac{1}{8}\int 1 - \cos(2x) - \cos^2(2x) + \cos^3(2x)\,dx$$

$$= \frac{1}{8}\left[\int 1\,dx - \int \cos(2x)\,dx - \int \cos^2(2x)\,dx + \int \cos^3(2x)\,dx\right]$$

$$= \frac{1}{8}\left[x - \frac{\sin(2x)}{2} - \right.$$

$\int \cos^3 x \sin^4 x\,dx$

odd power of $\cos x$
$u = \sin x$

$\int \cos^2 x \sin^4 x \cos x\,dx$

$\int \cos^3 x \sin^5 x\,dx$

$u = \cos x$

or

$u = \sin x$

Try it!

$\int \cos^3(2x)\,dx = ?$

Integrals involving powers of $\sec(x)$ and $\tan(x)$

Example 5. $\int \tan^6(x)\sec^4(x)\,dx$

even secant
$u = \tan x$
$du = \sec^2 x\,dx$

$\sec^2 x = 1 + \tan^2 x = 1 + u^2$

$$= \int \tan^6 x \sec^2 x \sec^2 x\,dx$$
$$= \int u^6 (1+u^2)\,du$$
$$= \int u^6 + u^8\,du$$
$$= \int u^6\,du + \int u^8\,du$$
$$= \frac{u^7}{7} + \frac{u^9}{9} + C = \boxed{\frac{\tan^7 u}{7} + \frac{\tan^9 u}{9} + C}$$

Example 6. $\int \tan^3(x)\sec(x)\,dx$

odd tangent
$u = \sec x$
$du = \sec x \tan x\,dx$

$\tan^2 x = \sec^2 x - 1 = u^2 - 1$

$$= \int \tan^2 x \sec x \tan x\,dx$$
$$= \int u^2 - 1\,du$$
$$= \frac{u^3}{3} - u + C$$
$$= \frac{\sec^3 x}{3} - \sec x + C$$

Example 7. $\int \sec^3(x)\,dx$ $= \int \sec^2 x \sec x\,dx$

$$= \int (1+\tan^2 x)\sec x\,dx$$

$\boxed{\sec^2 x = 1+\tan^2 x}$

$$= \underbrace{\int \sec x\,dx}_{} + \underbrace{\int \tan^2 x \sec x}_{}$$

$u = \sec x$

$du = \sec x \tan x\,dx$

$$= \int \sec x\,dx + \int \tan x \underbrace{\tan x \sec x\,dx}_{du}$$

?

Try it with IBP

$$\int \sec x \sec^2 x\,dx$$

$u = \sec x$

$du =$

$dv =$

$v = ?$

$u = \sec x$

$u^2 = \sec^2 x$

$\tan^2 x = \sec^2 x - 1$

$= u^2 - 1$

$\tan = \sqrt{u^2-1}$

Example 8. $\int \sin(4x)\cos(5x)\,dx$

Use $\sin(A)\cdot\cos(B) = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$

$\sin(4x)\cos(5x) = \frac{1}{2}\left(\sin(4x-5x) + \sin(4x+5x)\right)$

$= \frac{1}{2}\left[\sin(-x) + \sin(9x)\right]$

$= -\frac{1}{2}\sin(x) + \frac{1}{2}\sin(9x)$

we used that sine is an odd function

$\sin(-x) = -\sin x$

$$= \int -\frac{1}{2}\sin(x)\,dx + \frac{1}{2}\int \sin(9x)\,dx$$

$$= -\frac{1}{2}(-\cos x) + \frac{1}{2}\left(\frac{-\cos 9x}{9}\right)$$

$$= \boxed{\frac{1}{2}\cos x - \frac{1}{18}\cos 9(x) + C}$$