Math 140: Thursday Aug 29, Day 6 Notes

Agenda for Thursday, Aug 29,2024	
Hamiltonian Circuits and paths	**1**

Reminders

- **Online Homework 2 - Eulerization and Hamilton Circuits and Paths** due today by 11:59pm
- Office Hour **today 1-2:15pm** in C162
- Group Quiz 2 next Thursday
- Exam 1, two weeks from today

Hamiltonian Circuits and paths - continued

Recall:

What is a Hamiltonian circuit?

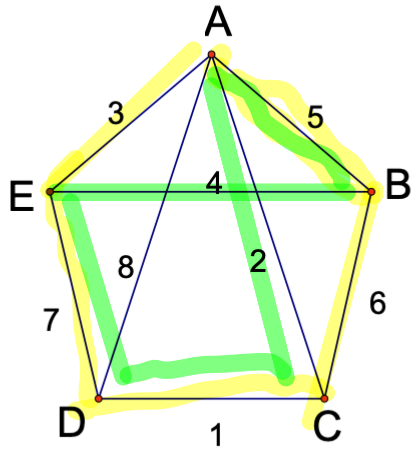
is a circuit that passes through all vertices exactly once.

What is a Hamiltonian path?

a pathot that visits every vertex exactly once.

From previous days notes:

Example 3. Consider the following graph.



a) What is the weight of AD? 8

b) What is the weight of the Hamilton Circuit ABCDEA?

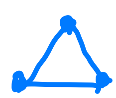
$5+6+1+7+3=22$

c) What is the weight of the Hamilton Circuit ACDEBA?

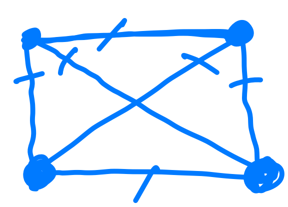
$2+1+8+4+5=19$

Complete graph A graph in which every pair of distinct vertices is joined by an edge. If you have N vertices, K_N denotes the complete graph.

$N=3$



K_3 edges $= 3$



K_4

edges $= 6$

Factorial:

we use ! → denotes factorial

$2! = 1 \cdot 2 = 2$

$3! = 1 \cdot 2 \cdot 3 = 6$

$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

$0! = 1$

! Formula for the number of edges in a complete graph with N vertices:

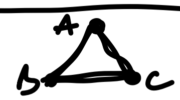
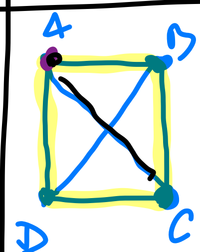
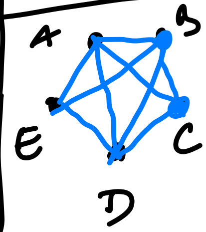
$$\frac{N(N-1)}{2}$$

ex: K_3 we have $N=3$

number of edges is in K_3 $\frac{3(3-1)}{2} = \frac{3\cdot 2}{2} = 3$

in K_4: $\frac{4(4-1)}{2} = \frac{4\cdot 3}{2} = 6$ edges

Table with info about complete graphs

Number of vertices	Complete graph K_N	Graph	Number of Hamiltonian circuits
$N=3$	K_3		ABCA ACBA 2
$N=4$	K_4		ABCDA or ADCB ACDBA or ABDCA ADBCA or ACBDA 6 
$N=5$	K_5		K_5 has 24 Hamiltonian circuits

! Formula for the number of Hamilton Circuits in the complete graph K_N:

K_3 : $2 = 2!$

K_4 : $6 = 3!$

K_5 : $24 = 4!$

K_N : $(N-1)!$ Hamiltonian circuits

Example 4. For a complete graph with 10 vertices, how many Hamilton Circuits are there?

$N=10$

$(10-1)! = 9! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9 = 362{,}880$

Example 5. If K_N has 40,320 Hamilton Circuits, what is N?

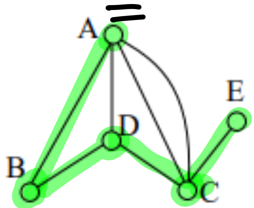
try on your own

hint: $\boxed{8! = 40{,}320}$

Warm up Question:

Does a Hamiltonian path or circuit exist on the graph below?

HP



no Hamiltonian Circuit

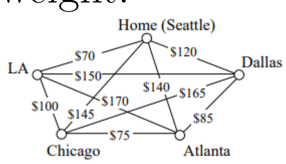
HC

yes HP: but it has to start or end at vertex E.

ex: ABDCE

ECABD

With Hamiltonian circuits, our focus will not be on existence, but on the question of optimization; given a graph where the edges have weights, can we find the optimal Hamiltonian circuit; the one with lowest total weight.



Brute Force Algorithm: (Exhaustive Search)

1. Make a list of all Hamilton Circuits from one starting vertex
2. Calculate the weight of each circuit.
3. Choose an optimal Circuit.

For K_4 you have to check 3! = 6 circuits.
For K_5 you have to check 4! = 24 circuits.
For K_6 you have to check 5! = 120 circuits.

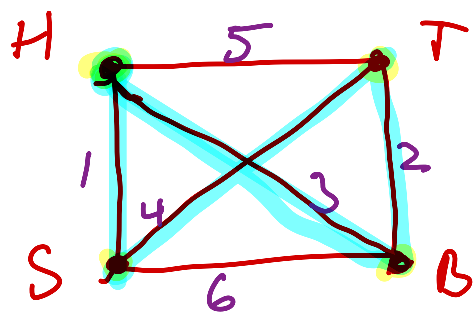
Is this an efficient algorithm? No

Brute Force is the only algorithm known at this time to always give an optimal circuit.

Example 1. You want to run errands around town. You want to start and end your errands at home and visit Target, the bank, and Starbucks, but not necessarily in that order. In the chart given below, the numbers represent miles between each destination.

	Home	Target	Bank	Starbucks
Home	*	5	3	1
Target	5	*	2	4
Bank	3	2	*	6
Starbucks	1	4	6	*

Draw a weighted graph from the information given in the chart.



List all of the possible Hamilton circuits and their weights.

HTBSH $5+2+6+1 = 14$ (mirror circuit HSBTH)

HBTSH $3+2+4+1 = 10$ (mirror HSTBH)

HTSBH $5+4+6+3 = 18$ (HBSTH)

What is the optimal circuit?

Approximate Algorithms

Nearest - Neighbor Algorithm: NNA

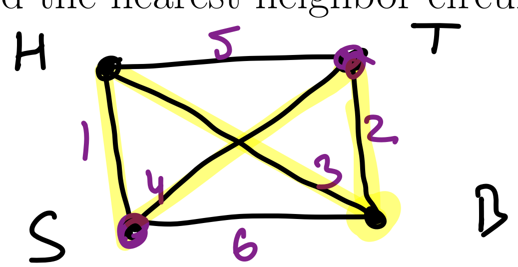
1. Pick a starting vertex
2. When you arrive at a vertex, go ot the nearest neighbor (go along the edge with the lowest weight) that has yet to be visited. Repeat to visit all vertices.
3. From the last vertex, return to the start.
4. Compute the weight of the circuit.
5. The circuit we obtain is called the **Nearest-Neighbor Circuit**.

NNC

Example 2. (again): You want to run errands around town. You want to start and end your errands at home and visit Target, the bank, and Starbucks, but not necessarily in that order. In the chart given below, the numbers represent miles between each destination.

	Home	Target	Bank	Starbucks
Home	*	5	3	1
Target	5	*	2	4
Bank	3	2	*	6
Starbucks	1	4	6	*

Find the nearest-neighbor circuit using the graph.



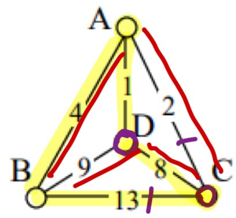
HSTBH → this is the NNC

1+4+2+3 = 10

it is the optimal circuit

~~Find the nearest-neighbor circuit using the chart.~~

Example 5. Apply the NNA algorithm to find a Hamiltonian circuit on the graph below.

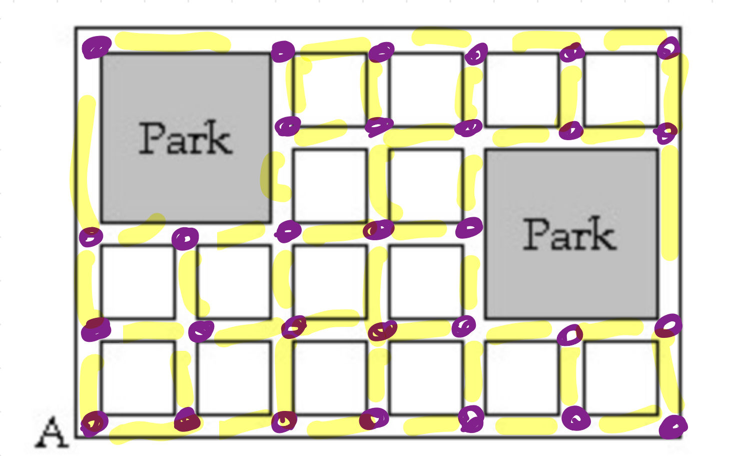


ADCBA
1+8+13+4 = 26

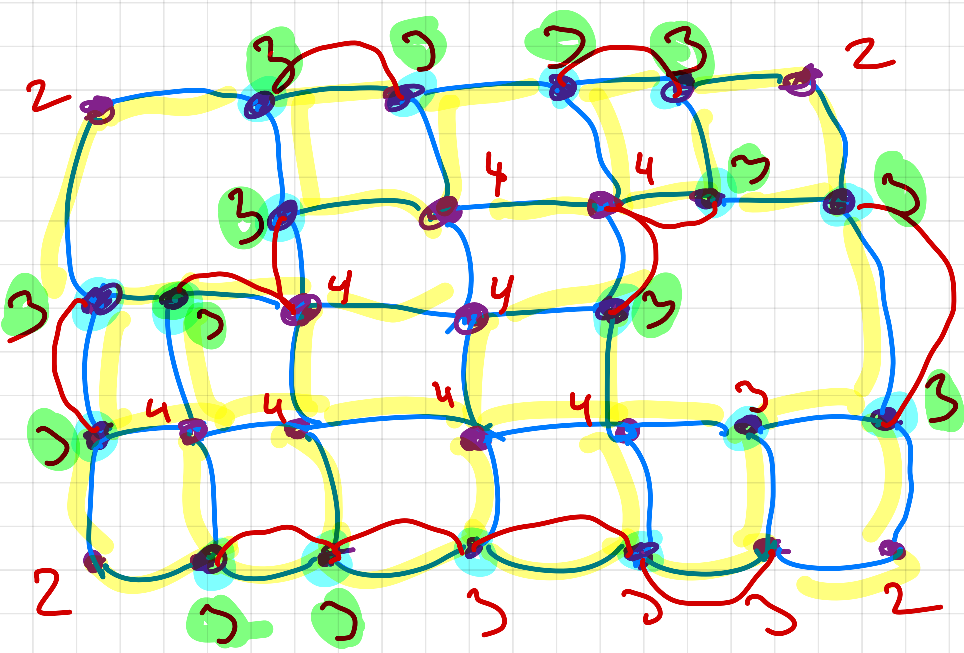
ACDBA
2+8+9+4 = 23

NNC is not the optimal circuit

Hint for question 1 on HW 2:



The corresponding graph is below.



The added red edges, are the edges necessary to Eulerize the graph.