Mth 146: Day 15
11.5 Alternating Series & Absolute Convergence
11.6 The Ratio and Root Tests

1 Part 1: New Concepts & Worked Examples

Recall: The Alternating Series Test

An **alternating series** is a series whose terms are alternately positive and negative. Its nth term has the form

$$a_n = (-1)^{n-1} b_n \qquad \text{or} \qquad a_n = (-1)^n b_n,$$

where $b_n > 0$ (in fact, $b_n = |a_n|$).

Alternating Series Test. If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - \cdots \qquad (b_n > 0)$$

satisfies the conditions

(i) _____ for all n

(ii) _____

then the series is _____.

Example 1. Show that the **alternating harmonic series**

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

is convergent.

The Alternating Series Estimation Theorem

Alternating Series Estimation Theorem. If $s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies

(i) $b_{n+1} \leq b_n$ and (ii) $\lim_{n \to \infty} b_n = 0$,

then

$$|R_n| = |s - s_n| \leq _____.$$

In words: the size of the error in using s_n to approximate s is no bigger than _____.

Note: This rule is valid *only* for alternating series that satisfy the conditions of the Alternating Series Test. It does not apply to other types of series.

Example 2. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

Absolute Convergence and Conditional Convergence

Given any series $\sum a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute values of the terms of the original series.

Definition 1. A series $\sum a_n$ is called **absolutely convergent** if _____.

Definition 2. A series $\sum a_n$ is called **conditionally convergent** if _____

_____; that is, if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Note: If $\sum a_n$ is a series with positive terms, then $|a_n| = a_n$, so absolute convergence is the same as convergence in this case.

Example 3. Show that the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ is absolutely convergent.

Example 4. Show that the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is conditionally convergent.

Theorem 3. If a series $\sum a_n$ is absolutely convergent, then it is _____.

Think of it this way: absolute convergence is a *stronger* type of convergence. An absolutely convergent series converges no matter what the signs of its terms are, while a conditionally convergent series (like the alternating harmonic series) would *not* converge if we made all of its negative terms positive.

The Ratio Test.

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is _____ (and therefore convergent).

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is _____.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the Ratio Test is _____; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Tip: The Ratio Test is usually conclusive when the nth term contains an **exponential** or a **factorial**.

Example 5. Test the series $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$ for absolute convergence.

Example 6. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

Example 7. Apply the Ratio Test to each of the following series. What do you notice?

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

Careful! When $L = 1$, the Ratio Test gives **no information**: the series could converge or diverge. The Ratio Test will *always* fail for p-series.

The Root Test.

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is _____ (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is _____.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is _____.

Tip: The Root Test is convenient to apply when n**th powers** occur. Also, if $L = 1$ in the Ratio Test, don't try the Root Test (and vice versa) — L will be 1 again.

Example 8. Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

Example 9. Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^n$ converges or diverges.

Lesson: When testing a series for convergence or divergence, it is often helpful to apply the **Test for Divergence** first, before attempting other tests.

2 Part 2: Now You Try It

Determine whether each series is **absolutely convergent**, **conditionally convergent**, or **divergent**. Final answers are given as footnotes at the bottom of the page — only the final answer, not the steps, so use them to check your work rather than replace it.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$[1]

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$[2]

[1] Answer: Absolutely convergent ($\sum 1/n^3$ is a p-series with $p = 3 > 1$).
[2] Answer: Conditionally convergent ($\sum 1/\sqrt[3]{n}$ diverges, $p = \frac{1}{3}$; the series converges by the AST).

3. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{2n+1}$[3]

4. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$[4]

[3]Answer: Divergent ($\lim a_n$ does not exist, so it diverges by the Test for Divergence).
[4]Answer: Absolutely convergent (Ratio Test, $L = 0 < 1$).

5. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n}{2^n n^3}$[5]

6. $\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$[6]

[5]Answer: Divergent (Ratio Test, $L=\frac{3}{2}>1$).
[6]Answer: Absolutely convergent (Root Test, $L=\frac{1}{2}<1$).

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$[7]

[7]Answer: Absolutely convergent (Root Test, $L = 0 < 1$).

3 Part 3: Reflection

For each item below, rate your comfort level and jot a quick note: What clicked? What's still fuzzy? What would help?

1. **Check the two conditions of the Alternating Series Test and use it to show an alternating series converges.** 1 2 3 4 5

2. **Use the Alternating Series Estimation Theorem to approximate the sum of a series to a given accuracy.** 1 2 3 4 5

3. **Classify a series as absolutely convergent, conditionally convergent, or divergent.** 1 2 3 4 5

4. **Apply the Ratio Test, including simplifying $|a_{n+1}/a_n|$ with factorials and exponentials.** 1 2 3 4 5

5. **Apply the Root Test, and recognize when the Ratio or Root Test is inconclusive ($L = 1$).** 1 2 3 4 5