Mth 140 Logic Review

Statements, Negation, Conjunction, and Disjunction

1 Statements

Definition: Statement

A statement (proposition) is a declarative sentence that is either true or false, but not both.

1.1 Examples of Statements

- "The sum of 2 and 3 is 5." (True statement)
- "Paris is the capital of Italy." (False statement)
- "All prime numbers are odd." (False statement)

1.2 Non-Examples (Not Statements)

- "What time is it?" (Question)
- "Please close the door." (Command)
- "x + 2 = 7" (Not a statement until x is specified)

Definition: Truth Value

The truth value of a statement is either true (T) or false (F).

2 Basic Logical Operations

2.1 Negation

Definition: Negation

The negation of a statement p, denoted by $\sim p$, is the statement "not p". The negation has the opposite truth value of the original statement.

Example: Negation

Let p: "Today is Monday"

- $\sim p$: "Today is not Monday"
- If p is true, then $\sim p$ is false
- If p is false, then $\sim p$ is true

Negation Truth Table

p	$\sim p$
T	F
F	T

3 Quantified Statements

Definition: Quantified Statement

A statement that expresses a property that is true for some or all elements in a given domain. The two main types are:

- Universal statements using words like: "all," "every," "for each," "any"
- Existential statements using words like: "some," "there exists," "there is," "at least one"

3.1 Universal Statements

Definition: Universal Statement

A statement claiming something is true for all elements in a domain.

Example: Universal Statements

- "All integers are divisible by 1"
- "Every student in the class passed the exam"
- "Any triangle has three sides"
- "Each real number has an opposite"

3.2 Existential Statements

Definition: Existential Statement

A statement claiming something is true for at least one element in a domain.

Example: Existential Statements

- "Some integers are even"
- "There is a student who got an A"
- "At least one person knows the answer"
- "There exists a number greater than zero"

3.3 Negating Quantified Statements

Rules for Negating Quantified Statements

Original Statement	Negation
"All P are Q" "Every P is Q"	"Some P are not Q" "At least one P is not Q"
"Some P are Q" "There exists a P that is Q"	"No P are Q" "All P are not Q"

Example: Negating Universal Statements

Original: "All cats are black"

- Negation: "Some cats are not black"
- Alternative: "At least one cat is not black"
- Alternative: "There is a cat that is not black"

Example: Negating Existential Statements

Original: "Some students passed the test"

- Negation: "No students passed the test"
- Alternative: "All students failed the test"

3.4 Common Quantified Statement Forms

Common Forms and Their Negations

- "All A are B"
  - Negation: "Some A are not B"
- "No A are B"
  - Negation: "Some A are B"
- "Some A are B"
  - Negation: "No A are B"
- "Some A are not B"
  - Negation: "All A are B"

3.5 Conjunction

Definition: Conjunction

The conjunction of statements p and q, denoted by $p \wedge q$, is the compound statement "p and q". It is true only when both p and q are true.

Example: Conjunction

Let p: "It is raining" and q: "It is cold"

- $p \wedge q$: "It is raining and it is cold"
- True only when both statements are true

Conjunction Truth Table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

3.6 Disjunction

Definition: Disjunction

The disjunction of statements p and q, denoted by $p \vee q$, is the compound statement "p or q". It is true when at least one of p or q is true.

Example: Disjunction

Let p: "I will study math" and q: "I will study history"

- $p \vee q$: "I will study math or I will study history"
- True when at least one statement is true

Disjunction Truth Table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

3.7 De Morgan's Laws

Definition: De Morgan's Laws

These laws show the relationship between negation, conjunction, and disjunction:

1. The negation of a conjunction: $\sim (p \wedge q) = \sim p \vee \sim q$
2. The negation of a disjunction: $\sim (p \vee q) = \sim p \wedge \sim q$

In words:

- "not (p and q)" = "not p or not q"
- "not (p or q)" = "not p and not q"

Example: De Morgan's First Law

Let p: "It is sunny" and q: "It is warm"
Consider $\sim (p \wedge q)$: "It is not (sunny and warm)"
This is equivalent to $\sim p \vee \sim q$: "It is not sunny or it is not warm"
Both statements are true when it's not both sunny and warm.

Example: De Morgan's Second Law

Let p: "I will study math" and q: "I will study physics"
Consider $\sim (p \vee q)$: "It is not true that (I will study math or physics)"
This is equivalent to $\sim p \wedge \sim q$: "I will not study math and I will not study physics"
Both statements mean I will study neither subject.

Truth Table for First De Morgan's Law

p	q	$p \wedge q$	$\sim (p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Truth Table for Second De Morgan's Law

p	q	$p \vee q$	$\sim (p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

4 Practice Problems

1. Determine which of the following are statements:

- "The square root of 16 is 4."
- "What is your name?"
- "x is greater than 5."
- "Let's go to the movies!"

2. Write the negation of each statement:

- "All dogs are brown"
- "The temperature is below 32°F"
- "The sun is shining or it is raining"

3. Given the statements:

- p: "The car is red"
- q: "The car is new"

Write in words:

- $p \wedge q$
- $p \vee q$
- $\sim p \wedge q$
- $\sim (p \vee q)$

4. Complete the following truth table:

p	q	$\sim p$	$p \wedge q$	$p \vee \sim q$
T	T			
T	F			
F	T			
F	F			

5. Use De Morgan's Laws to write the negation of each statement:

- "The movie is long and boring"
- "I will go to the beach or the park"
- "The number is positive and even"

6. For each statement, determine if they are equivalent using truth tables:

- $\sim (p \wedge q)$ and $\sim p \vee \sim q$
- $\sim (p \vee q)$ and $\sim p \wedge \sim q$

7. Write the negation of each quantified statement:

- "All prime numbers are odd"

- "Some students like mathematics"
- "No dogs can fly"
- "There exists a number that is both prime and even"

8. Determine if each statement is universal or existential:
   - "Every natural number has a successor"
   - "There is a largest prime number"
   - "Some polynomials have no real roots"
   - "All triangles have three sides"

Solutions to Practice Problems

Problem 1 Solutions

Determine which of the following are statements:

- "The square root of 16 is 4."
  Solution: This IS a statement (True)
- "What is your name?"
  Solution: This is NOT a statement (Question)
- "x is greater than 5."
  Solution: This is NOT a statement (Variable not specified)
- "Let's go to the movies!"
  Solution: This is NOT a statement (Command/Exclamation)

Problem 2 Solutions

Write the negation of each statement:

- "All dogs are brown"
  Solution: "Some dogs are not brown" or "There is a dog that is not brown"
- "The temperature is below 32°F"
  Solution: "The temperature is not below 32°F" or "The temperature is at or above 32°F"
- "The sun is shining or it is raining"
  Solution: "The sun is not shining and it is not raining"

Problem 3 Solutions

Given p: "The car is red" and q: "The car is new", write in words:

- $p \wedge q$
  Solution: "The car is red and new"
- $p \vee q$
  Solution: "The car is red or new (or both)"
- $\sim p \wedge q$
  Solution: "The car is not red and is new"
- $\sim (p \vee q)$
  Solution: "The car is neither red nor new"

Problem 4 Solution

Complete the truth table:

p	q	$\sim p$	$p \wedge q$	$p \vee \sim q$
T	T	F	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	T

Problem 7 Solutions

Write the negation of each quantified statement:

- "All prime numbers are odd"
  Solution: "Some prime numbers are not odd" or "There is a prime number that is even"
- "Some students like mathematics"
  Solution: "No students like mathematics" or "All students dislike mathematics"
- "No dogs can fly"
  Solution: "Some dogs can fly" or "There is a dog that can fly"
- "There is a number that is both even and prime"
  Solution: "No numbers are both even and prime" or "All numbers that are even are not prime"

Problem 8 Solutions

Determine if each statement is universal or existential:

- "Every natural number has a next number"
  Solution: Universal (uses "every")
- "There is a largest whole number"
  Solution: Existential (uses "there is")
- "Some triangles are right triangles"
  Solution: Existential (uses "some")
- "All squares are rectangles"
  Solution: Universal (uses "all")