Using Sum and Difference Formulas

The sum and difference formulas allow us to express the sine, cosine, and tangent of a sum or difference of angles in terms of the sine, cosine, and tangent of the individual angles. These formulas are particularly useful for evaluating trigonometric expressions, solving equations, and proving identities.

$$
\begin{aligned}
\text{I}\quad \sin(u+v) &= \sin u\cos v + \cos u \sin v \\
\text{II}\quad \sin(u-v) &= \sin u\cos v - \cos u \sin v \\
\text{III}\quad \cos(u+v) &= \cos u\cos v - \sin u \sin v \\
\text{IV}\quad \cos(u-v) &= \cos u\cos v + \sin u \sin v \\
\tan(u+v) &= \frac{\tan u + \tan v}{1-\tan u\tan v} \\
\tan(u-v) &= \frac{\tan u - \tan v}{1+\tan u\tan v}
\end{aligned}
$$

$\frac{4}{4}\frac{\pi}{3} - \frac{\pi}{4}\frac{3}{3} = \frac{4\pi - 3\pi}{12}$

$30°, 45°, 60°, 90°$

Example 1 Evaluating a Trigonometric Function

$\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$

Find the exact value of

1. $\cos 75°$

$\boxed{75° = 30° + 45°}$

$\boxed{\cos(75°)} = \cos(\underset{u}{30°} + \underset{v}{45°})$ III

$= \cos 30° \cos 45° - \sin 30° \sin 45°$

$= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} - \frac{1}{2}\frac{\sqrt{2}}{2}$

$= \boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$

2. $\sin\frac{\pi}{12} = \sin\left(\underset{u}{\frac{\pi}{3}} - \underset{v}{\frac{\pi}{4}}\right)$ II

$= \sin\frac{\pi}{3}\cdot\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\cdot\sin\frac{\pi}{4}$

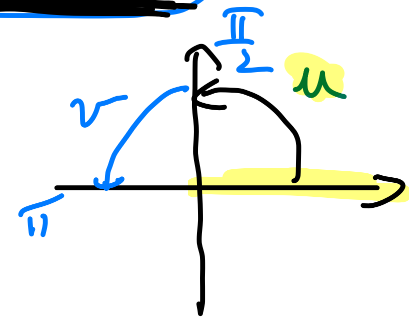
$= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2}$

$= \frac{\sqrt{6}-\sqrt{2}}{4}$

Example 2 Evaluating a Trigonometric Expression

Find the exact value of $\sin(u+v)$ given $\sin u = 4/5$, where $0 < u < \pi/2$, and $\cos v = -12/13$, where $\pi/2 < v < \pi$.

Solution:



$$\boxed{\sin(u+v)} = \sin u \cos v + \cos u \sin v$$

$$= \frac{4}{5}\left(-\frac{12}{13}\right) + \cos u \sin v$$

$$= \frac{-48}{65} + \frac{3}{5} \cdot \frac{5}{13} = \frac{-48+15}{65}$$

$$= \boxed{\frac{-33}{65}}$$

$$\sin^2 v + \cos^2 v = 1$$

$$\sin^2 v + \left(-\frac{12}{13}\right)^2 = 1$$

$$\sin^2 v = 1 - \frac{144}{169}$$

$$\sin^2 v = \frac{169}{169} - \frac{144}{169}$$

$$\sin^2 v = \frac{25}{169} \Rightarrow \sin v = \pm\frac{5}{13}$$

$$\sin v = \frac{5}{13}$$

$$\sin^2 u + \cos^2 u = 1$$

$$\left(\frac{4}{5}\right)^2 + \cos^2 u = 1$$

$$\cos^2 u = 1 - \frac{16}{25}$$

$$\cos^2 u = \frac{25}{25} - \frac{16}{25}$$

$$\cos^2 u = \frac{9}{25}$$

$$\cos u = \pm\frac{3}{5}$$

$$\cos u = \frac{3}{5}$$

Example 4 **Proving a Cofunction Identity**

$\cos(u-v) = \cos u \cos v + \sin u \sin v$

Prove the cofunction identity $\cos\left(\frac{\pi}{2} - x\right) = \sin x$.
Solution:

$$\boxed{\cos\left(\underset{u}{\frac{\pi}{2}} - \underset{v}{x}\right)} = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\cdot\sin x$$

$$= 0\cdot\cos x + 1\cdot\sin x$$

$$= 0 + \sin x$$

$$\boxed{= \sin x}$$

Example 5 **Solving a Trigonometric Equation**

$\sin(u+v) = \sin u \cos v + \cos u \sin v$
$\sin(u-v) = \qquad\qquad -$

Find all solutions of

$$\sin\left(\underset{u}{x} + \underset{v}{\frac{\pi}{4}}\right) + \sin\left(x - \frac{\pi}{4}\right) = -1$$

in the interval $[0, 2\pi)$.
Solution:

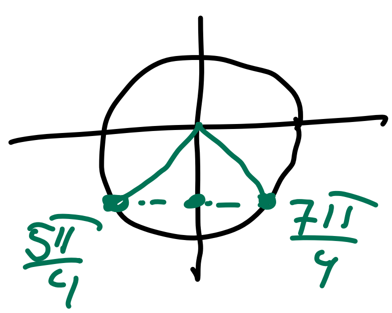
$$\sin x \cos\frac{\pi}{4} + \cancel{\cos x \sin\frac{\pi}{4}} + \sin x \cos\frac{\pi}{4} - \cancel{\cos x \sin\frac{\pi}{4}} = -1$$

$$2\sin \cos\pi/4 = -1$$

$$\cancel{2}\sin x\left(\frac{\sqrt{2}}{\cancel{2}}\right) = -1$$

$$\sin x = -\frac{1}{\sqrt{2}}$$

$$\boxed{\sin x = -\frac{\sqrt{2}}{2}}$$



Ans $x = \frac{5\pi}{4}$ & $x = \frac{7\pi}{4}$

Example 7 An Application from Calculus

Verify that $\frac{\sin(x+h)-\sin x}{h} = (\cos x)\left(\frac{\sin h}{h}\right) - (\sin x)\left(\frac{1-\cos h}{h}\right)$, where $h \neq 0$.

Solution:

Recall: Difference quotient for $f(x)$

$\frac{f(x+h)-f(x)}{h}$ $\leadsto$ in the limit as h goes 0 — derivative of f

$f(x) = \sin x$

Diff quotien

$$* \quad \frac{\sin(\overset{u}{x}+\overset{v}{h}) - \sin x}{h} =$$

$$= \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} =$$

$$= \frac{\sin x \cos h}{h} + \frac{\cos x \sin h}{h} - \frac{\sin x}{h}$$

$$= \cos x \frac{\sin h}{h} - \sin x \left(1 - \frac{\cos h}{h}\right)$$

1 0

Note: This identity is particularly important in calculus as it helps establish the derivative of the sine function. As h approaches 0, $\frac{\sin h}{h}$ approaches 1 and $\frac{1-\cos h}{h}$ approaches 0, which leads to the result that $\frac{d}{dx}(\sin x) = \cos x$.

0