Mth 146: Day 20
11.11 Approximating Functions by Taylor Polynomials
(Solutions)

1 Part 1: New Concepts & Worked Examples

Taylor Polynomials as Approximations

Suppose f is equal to the sum of its Taylor series at a. In *Taylor and Maclaurin Series* we called the nth partial sum of that series the **nth-degree Taylor polynomial of f at a**:

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$
$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Since $T_n(x) \to f(x)$ as $n \to \infty$, we can use T_n as an approximation:

$$f(x) \approx T_n(x)$$

Teaching note: Start by connecting to last class: a Taylor series is an infinite sum, and in practice we can only add finitely many terms. T_n is exactly "stop after the $(x-a)^n$ term." Point out that the summation

index is i (not n) because n is now fixed: it is the *degree*. Students often think T_n has n terms; it has up to $n+1$ terms (from $i=0$ to $i=n$), and some can be 0. For example, for $\sin x$ at 0, $T_3 = T_4 = x - \frac{x^3}{6}$ because the x^4 coefficient is 0.

Key facts:

- The first-degree Taylor polynomial $T_1(x) = f(a) + f'(a)(x-a)$ is the same as the **linearization** of f at a (the tangent line).
- **T_n and f have the same derivatives at a** up to and including order n.
- **The farther x is from a, the slower $T_n(x)$ converges to $f(x)$**, so we need a higher degree to get the same accuracy.

Teaching note: Why does matching derivatives make a good approximation? Build it up: T_0 matches the *value* $f(a)$ (a horizontal line). T_1 also matches the *slope* $f'(a)$ (the tangent line from Calc 1 linear approximation). T_2 also matches the *concavity* $f''(a)$, so it bends the same way as f. Each extra derivative makes the polynomial "hug" the curve a little longer as you move away from a. A quick sketch of e^x with T_1 and T_2 at 0 makes this very visual.

For $f(x) = e^x$ at $a = 0$, compare the Taylor polynomials with the true value:

	$x = 0.2$	$x = 3.0$
$T_2(x)$	1.220000	8.500000
$T_4(x)$	1.221400	16.375000
$T_6(x)$	1.221403	19.412500
$T_8(x)$	1.221403	20.009152
$T_{10}(x)$	1.221403	20.079665
e^x	1.221403	20.085537

At $x = 0.2$ (close to 0) the convergence is very fast; at $x = 3$ (far from 0) it is much slower.

Teaching note: Have students read the table column by column. At $x = 0.2$, T_6 already agrees with $e^{0.2}$ to six decimal places. At $x = 3$, even T_{10} is still off in the second decimal place. Ask: *why?* The next term is $\frac{x^{n+1}}{(n+1)!}$. When $|x| < 1$, the power x^{n+1} shrinks *and* the factorial grows, so the terms die fast. When $x = 3$, the power 3^{n+1} grows, and the factorial needs many more terms before it wins. This is the main idea of the day: **accuracy depends on both n and how far x is from a.**

Important notes:

- T_n is the nth partial sum of the Taylor series, so near a we use $f(x) \approx T_n(x)$.
- T_1 is the tangent line; T_2 also matches the concavity; each higher degree matches one more derivative.
- The approximation gets better when n is larger **or** when x is closer to a.

How Good Is the Approximation?

When we use T_n to approximate f, we want to know: **How big is the error? How large must n be to get a desired accuracy?** The error is the absolute value of the remainder:

$$|R_n(x)| = |f(x) - T_n(x)|$$

Three ways to estimate the error:

1. Use a calculator or computer to **graph** $|R_n(x)|$.
2. If the series is **alternating**, use the **Alternating Series Estimation Theorem**.
3. In **all** cases, we can use **Taylor's Inequality**.

Teaching note: Stress the logic: if we knew $f(x)$ exactly, we would not need an approximation! So we cannot compute $R_n(x)$ exactly. Instead we find an **upper bound**: a number we can guarantee the error is smaller than. Words to use: "accurate to within," "the error is at most," "the maximum possible error." Both theorems below were introduced earlier; today we use them on Taylor polynomials.

Recall the two error tools:

Taylor's Inequality. If $\left|f^{(n+1)}(x)\right| \leq M$ for $|x-a| \leq d$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \leq d$$

Teaching note: Point at each piece of Taylor's Inequality. It looks just like the *next term* of the Taylor series, $\frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1}$, except $f^{(n+1)}(a)$ is replaced by M, the **biggest** $|f^{(n+1)}|$ can be on the whole interval. Two common mistakes: (1) using the nth derivative instead of the $(n+1)$st, and (2) evaluating M only at $x=a$ instead of finding the maximum over the interval.

Alternating Series Estimation Theorem. If $\sum(-1)^{n-1}b_n$ satisfies $b_{n+1} \leq b_n$ and $b_n \to 0$, then the error from stopping early is at most

$$|R_n| \leq b_{n+1} \qquad \textbf{(the first term we left out)}$$

Example 1. (a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.
(b) How accurate is this approximation when $7 \le x \le 9$?

> **Teaching note:** Before computing, ask the class: *why center at $a = 8$?* Because 8 is a perfect cube ($\sqrt[3]{8} = 2$), so $f(8)$, $f'(8)$, and $f''(8)$ come out as nice fractions, and 8 is in the middle of the interval $7 \le x \le 9$. **Rule of thumb: center at a number close to the x-values you care about where f and its derivatives are easy to evaluate.**

Solution: (a) Compute derivatives and evaluate at 8:

$$\begin{aligned} f(x) &= \sqrt[3]{x} = x^{1/3} & f(8) &= 2 \\ f'(x) &= \tfrac{1}{3}x^{-2/3} & f'(8) &= \tfrac{1}{12} \\ f''(x) &= -\tfrac{2}{9}x^{-5/3} & f''(8) &= -\tfrac{1}{144} \\ f'''(x) &= \tfrac{10}{27}x^{-8/3} \end{aligned}$$

So the second-degree Taylor polynomial is

$$\begin{aligned} T_2(x) &= f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2 \\ &= 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 \end{aligned}$$

and the approximation is $\sqrt[3]{x} \approx 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$.
(b) The series is **not alternating** when $x < 8$, so we use **Taylor's Inequality** with $n = 2$ and $a = 8$:

$$|R_2(x)| \le \frac{M}{3!}|x-8|^3 \qquad \text{where } |f'''(x)| \le M.$$

Because $x \ge 7$, we have $x^{8/3} \ge 7^{8/3}$, so

$$f'''(x) = \frac{10}{27} \cdot \frac{1}{x^{8/3}} \le \frac{10}{27} \cdot \frac{1}{7^{8/3}} < 0.0021.$$

Take $M = 0.0021$. Also $7 \le x \le 9$ means $-1 \le x - 8 \le 1$, so $|x - 8| \le 1$. Then

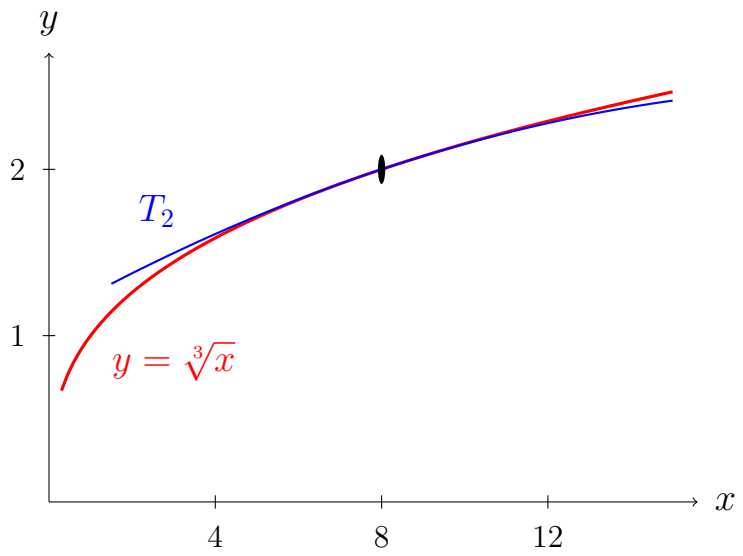
$$|R_2(x)| \le \frac{0.0021}{3!} \cdot 1^3 = \frac{0.0021}{6} < 0.0004.$$

So for $7 \le x \le 9$ the approximation is **accurate to within 0.0004**.

Teaching note: Things to say while working part (a): the f''' line has no value at 8 because T_2 only needs up to f'', but we compute f''' now since part (b) needs it. In T_2, the coefficient $-\frac{1}{144} \div 2! = -\frac{1}{288}$; students often forget to divide by 2!.

Teaching note: Part (b), step by step. **Why not alternating?** For $n \ge 1$ the derivatives $f^{(n)}(8)$ alternate in sign $(+, -, +, \ldots)$. When $x < 8$, $(x-8)^n$ *also* alternates in sign, so the signs cancel and every term after the first has the same sign. That means the Alternating Series Estimation Theorem does not apply on the whole interval. **Finding M:** $f'''(x) = \frac{10}{27}x^{-8/3}$ is *decreasing*, so its largest value on $[7, 9]$ is at the left endpoint $x = 7$. On a calculator, $7^{8/3} \approx 179.3$ and $\frac{10}{27} \div 179.3 \approx 0.00207$; we round *up* to 0.0021 so M is still a valid upper bound. **Finding $|x - a|$:** the farthest x gets from 8 is 1.

Teaching note: Sanity check to show the class: $T_2(7) = 2 - \frac{1}{12} - \frac{1}{288} \approx 1.913194$ and $\sqrt[3]{7} \approx 1.912931$, so the actual error is about 0.00026. At $x = 9$, $T_2(9) \approx 2.079861$ and $\sqrt[3]{9} \approx 2.080084$, error about 0.00022. Both are below the guaranteed bound 0.0004, as they must be.



$y = \sqrt[3]{x}$ (red) and $y = T_2(x)$ (blue) are very close near $x = 8$

Checking with a graph: graphing $|R_2(x)| = |\sqrt[3]{x} - T_2(x)|$ on $7 \le x \le 9$ shows $|R_2(x)| < 0.0003$. So the graph gives a slightly better estimate than Taylor's Inequality. **Taylor's Inequality gives an upper bound, not the exact error.**

Teaching note: If you have Desmos open, graph $y = \left|\sqrt[3]{x} - \left(2 + \frac{x-8}{12} - \frac{(x-8)^2}{288}\right)\right|$ on $7 \le x \le 9$. The graph touches 0 at $x = 8$ (no error at the center) and grows toward the endpoints. Its highest point is about 0.00026 at $x = 7$, matching the sanity check above. Taylor's Inequality is *safe* but not *tight*: it uses the worst-case value of f''' everywhere on the interval.

Important notes:

Steps for Taylor's Inequality:

1. Find the $(n+1)$st derivative $f^{(n+1)}(x)$ (one more than the degree).
2. Find $M =$ the largest value of $\left|f^{(n+1)}(x)\right|$ on the interval (often at an endpoint). Round **up**.
3. Find $d =$ the largest value of $|x-a|$ on the interval.
4. Plug in: $|R_n(x)| \leq \dfrac{M}{(n+1)!}\, d^{n+1}$.

Summary: Which error method should I use?

- Series is alternating (with decreasing terms) $\Rightarrow$ Alternating Series Estimation Theorem: error $\leq$ first omitted term.
- Series is not alternating (or you are not sure) $\Rightarrow$ Taylor's Inequality: find $M \geq \left|f^{(n+1)}\right|$ on the interval.
- Want to check or improve the estimate $\Rightarrow$ graph $|R_n(x)|$ with technology.

Teaching note: Give a quick example of the alternating case so students see the contrast with Example 1: for $\sin x \approx x - \frac{x^3}{3!}$ with $|x| \leq 0.3$, the Maclaurin series of $\sin x$ alternates, so the error is at most the first omitted term, $\frac{|x|^5}{5!} \leq \frac{(0.3)^5}{120} \approx 2 \times 10^{-5}$. No M needed!

Why this matters: when you press the sin or e^x key on a calculator, the machine often computes a **polynomial approximation**, usually a modified Taylor polynomial.

Important notes:

- The error is $|R_n(x)| = |f(x) - T_n(x)|$. We find an **upper bound** for it, not its exact value.
- Alternating series: error $\leq$ the first term left out. Otherwise: Taylor's Inequality.
- Choose the center a close to the x-values you need, where f and its derivatives are easy to compute.