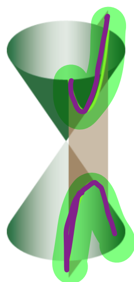
1. Hyperbolas

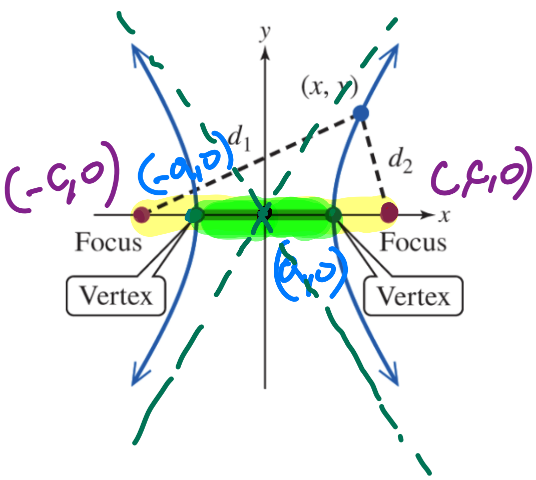
In this section we study hyperbolas from a geometric point of view. We begin with the geometric definition of a hyperbola. Although ellipses and hyperbolas have completely different shapes, their definitions and equations are similar.
Instead of using the sum of distances from two fixed foci, as in the case of an ellipse, we use the difference to define a hyperbola.



Hyperbolas have 2 branches

GEOMETRIC DEFINITION OF A HYPERBOLA

A **hyperbola** is the set of all points in the plane, the difference of whose distances from two fixed points F_1 and F_2 is a constant. (See Figure 1.) These two fixed points are the **foci** of the hyperbola.



orange : transverse axis with length 2a

2. Equations and Graphs of hyperbolas

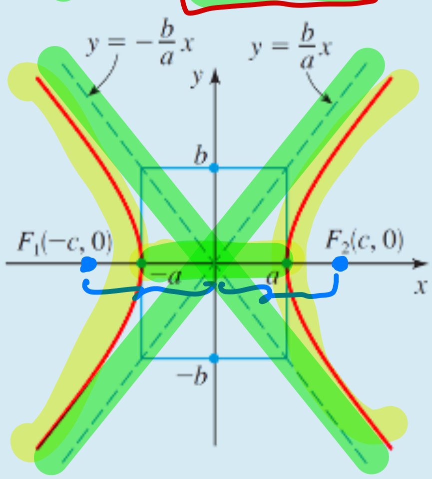
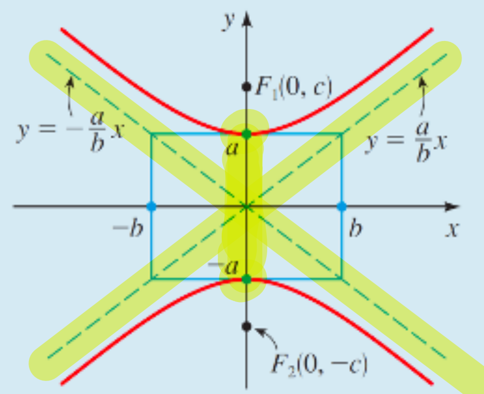
The following box summarizes the equation and features of hyperbolas centered at the origin.

HYPERBOLA WITH CENTER AT THE ORIGIN

positive term is with x^2

positive term is with the y^2

The graph of each of the following equations is a hyperbola with center at the origin and having the given properties.

EQUATION	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad a > 0, b > 0$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad a > 0, b > 0$
VERTICES	$(\pm a, 0)$	$(0, \pm a)$
TRANSVERSE AXIS	Horizontal, length $2a$	Vertical, length $2a$
ASYMPTOTES	$y = \pm \frac{b}{a}x$	$y = \pm \frac{a}{b}x$
FOCI	$(\pm c, 0)$, $\boxed{c^2 = a^2 + b^2}$	$(0, \pm c)$, $\boxed{c^2 = a^2 + b^2}$
GRAPH		

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

3. Graphing hyperbolas

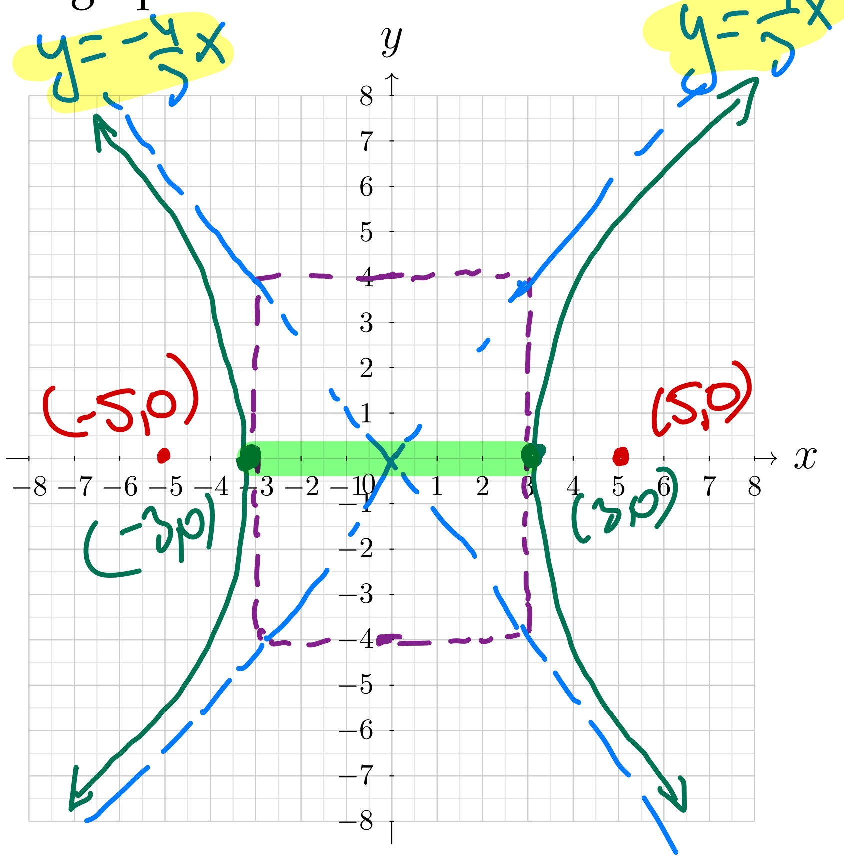
The following box summarizes the equation and features of hyperbolas centered at the origin.

HOW TO SKETCH A HYPERBOLA

→ refrence rectangle

1. **Sketch the Central Box.** This is the rectangle centered at the origin, with sides parallel to the axes, that crosses one axis at $\pm a$ and the other at $\pm b$.
2. **Sketch the Asymptotes.** These are the lines obtained by extending the diagonals of the central box.
3. **Plot the Vertices.** These are the two x-intercepts or the two y-intercepts.
4. **Sketch the Hyperbola.** Start at a vertex, and sketch a branch of the hyperbola, approaching the asymptotes. Sketch the other branch in the same way.

Example 1: A hyperbola has the equation $16x^2 - 9y^2 = 144$. Find the vertices, foci, length of the transverse axis, and asymptotes, and sketch the graph.



Divide by 144

$\frac{16x^2}{144} - \frac{9y^2}{144} = \frac{144}{144}$

$\frac{x^2}{9} - \frac{y^2}{16} = 1$

$\boxed{\frac{x^2}{3^2} - \frac{y^2}{4^2} = 1}$ *

$a = 3$, $b = 4$

Transverse axis: $2a = 2 \cdot 3$ = 6

foci: $(c,0)$ & $(-c,0)$

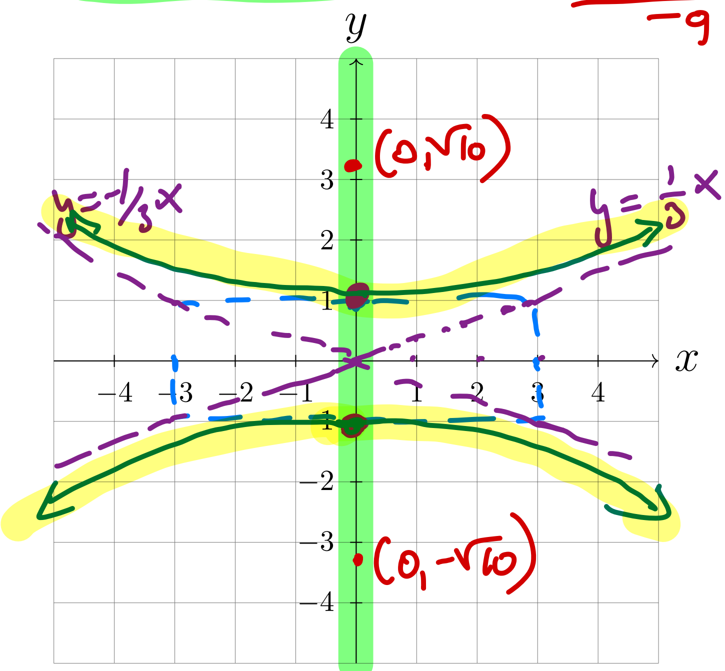
$c^2 = a^2 + b^2 = 9 + 16 = 25$

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ or $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$

Example 2: Find the vertices, foci, length of the transverse axis, and asymptotes of the hyperbola, and sketch its graph.

$x^2 - 9y^2 + 9 = 0$ $\Rightarrow \frac{x^2-9y^2}{-9}=\frac{-9}{-9} \Rightarrow \frac{x^2}{-9}+y^2=1$

$\Rightarrow y^2-\frac{x^2}{9}=1 \Rightarrow \frac{y^2}{1^2}-\frac{x^2}{3^2}=1$



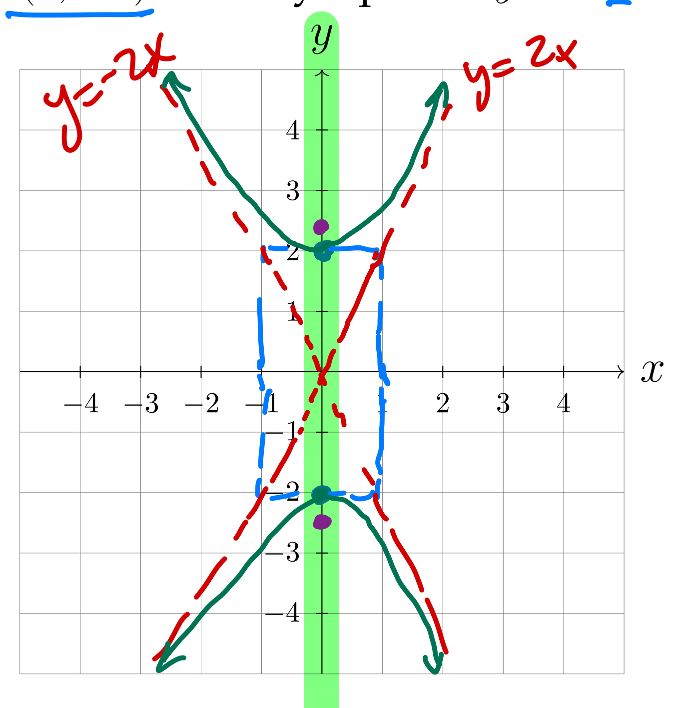
$\boxed{a=1}, b=3$

transverse axis: $2a=2\cdot1=2$

foci: $c^2=a^2+b^2=1+9=\sqrt{10}$

$(0,\sqrt{10}),(0,-\sqrt{10})$

Example 3: Find the equation and the foci of the hyperbola with vertices $(0, \pm 2)$ and asymptotes $y = \pm 2x$. Sketch the graph.



$\boxed{\frac{y^2}{a^2}-\frac{x^2}{b^2}=1}$

$\left.\begin{array}{l} y=\pm\frac{a}{b}x \\ y=\pm\frac{2}{1}x \end{array}\right\}$ $\begin{array}{l} a=2 \\ b=1 \end{array}$

foci: $c^2=a^2+b^2$
$=4+1=\sqrt{5}$

equation: $\frac{y^2}{2^2}-\frac{x^2}{1}=1$

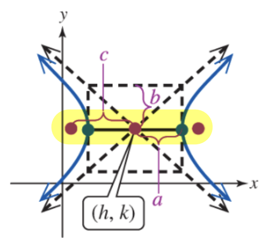
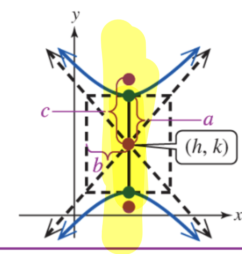
$\frac{y^2}{4}-x^2=1$

4. Hyperbolas with center at (h, k)

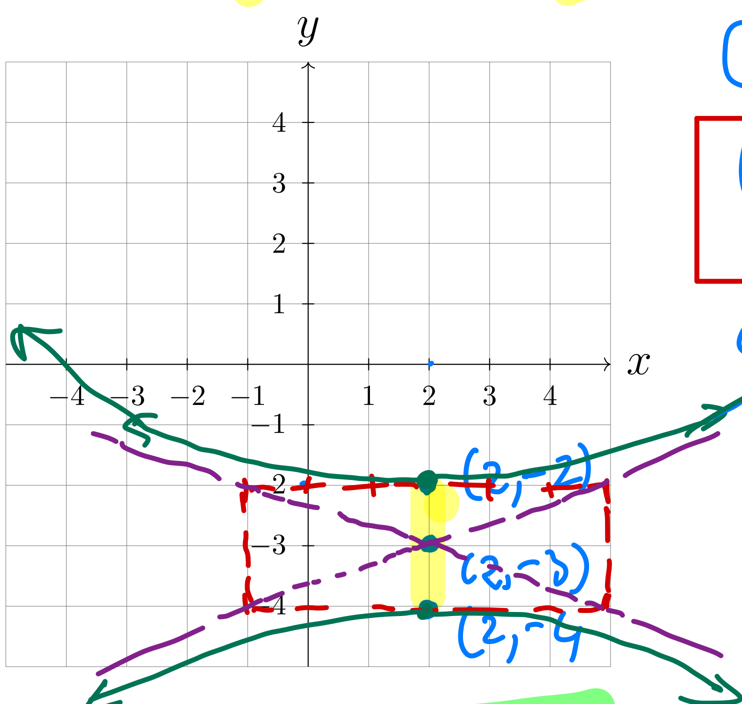
The following box summarizes the equation and features of hyperbolas centered at (h, k).

Standard Forms of an Equation of a Hyperbola Centered at (h, k)

The standard forms of an equation of a hyperbola centered at (h, k) are as follows. Assume that $a > 0$ and $b > 0$.

	Transverse Axis Horizontal	Transverse Axis Vertical
Equation:	$\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$	$\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$
Center:	(h, k)	(h, k)
Foci: (*Note*: $c^2 = a^2 + b^2$)	$(h+c, k)$ and $(h-c, k)$	$(h, k+c)$ and $(h, k-c)$
Vertices:	$(h+a, k)$ and $(h-a, k)$	$(h, k+a)$ and $(h, k-a)$
Asymptotes:	$y-k = \pm\dfrac{b}{a}(x-h)$	$y-k = \pm\dfrac{a}{b}(x-h)$
Graph:		

Example 4: Given $-\dfrac{(x-2)^2}{9} + (y+3)^2 = 1$, graph the hyperbola and identify the center, vertices, foci and equations of the asymptotes.



$(y+3)^2 - \dfrac{(x-2)^2}{9} = 1$

$\boxed{\dfrac{(y-(-3))^2}{1^2} - \dfrac{(x-2)^2}{3^2} = 1}$

center: $(2,-3)$

$a = 1$, $b = 3$

$c^2 = a^2 + b^2 = 1 + 9 = 10$

$c = \sqrt{10}$

vertex: $(2,-2)$

$y-k = \pm\frac{a}{b}(x-h)$

$y-(-3) = \pm\frac{1}{3}(x-2)$

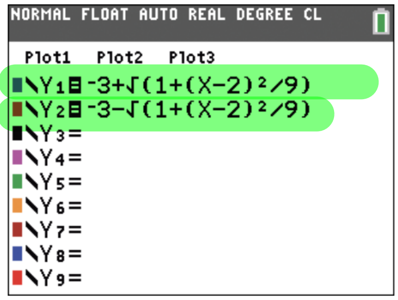
Calculator Instructions:

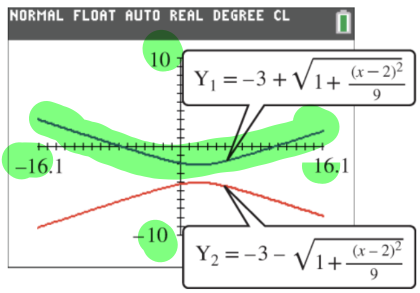
$$(y+3)^2 - \frac{(x-2)^2}{9} = 1$$

$$(y+3)^2 = 1 + \frac{(x-2)^2}{9}$$

$$y+3 = \pm\sqrt{1+\frac{(x-2)^2}{9}}$$

$$y = -3 \pm \sqrt{1+\frac{(x-2)^2}{9}}$$





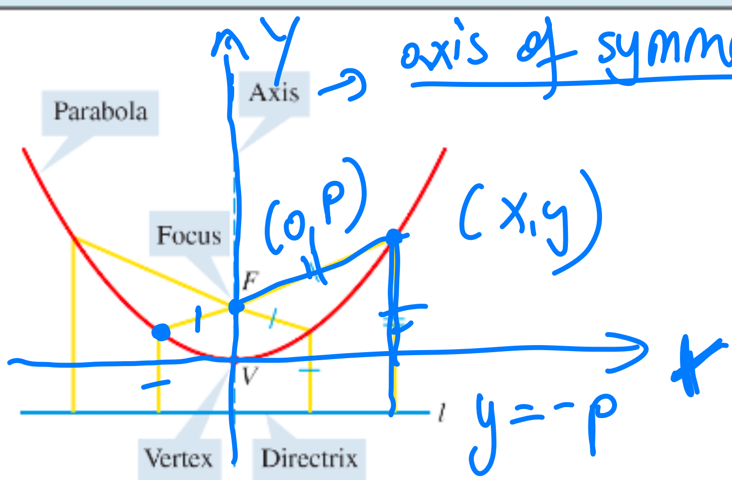
5. 9.1 Continued: Parabolas

In this section we study parabolas from a geometric rather than an algebraic point of view.
We begin with the geometric definition of a parabola and show how this leads to the algebraic formula that we are already familiar with.

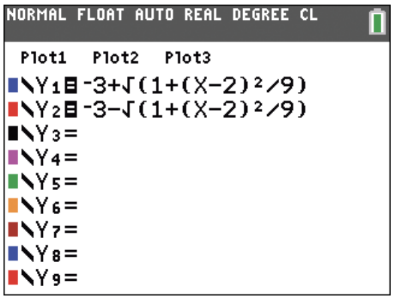
GEOMETRIC DEFINITION OF A PARABOLA

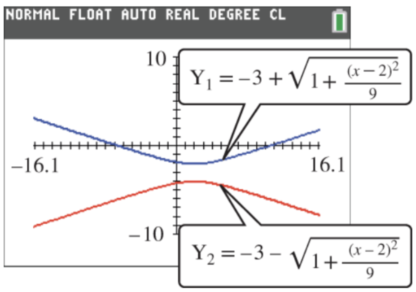
A **parabola** is the set of all points in the plane that are equidistant from a fixed point F (called the **focus**) and a fixed line l (called the **directrix**).

axis of symmetry vertical in this case

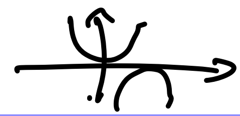


Calculator Instructions:





$y = ax^2 + bx + c$

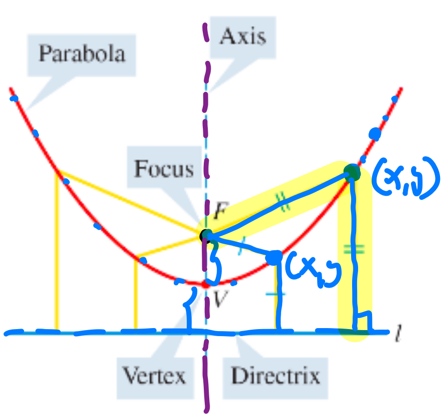


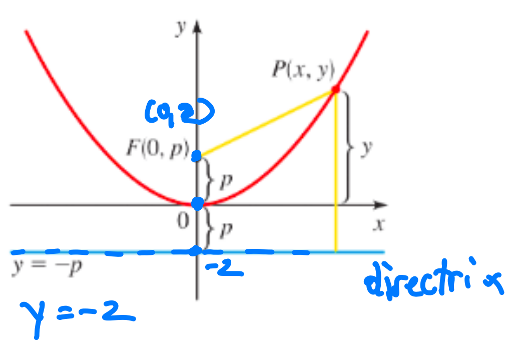
5. 9.1 Continued: Parabolas

In this section we study parabolas from a geometric rather than an algebraic point of view. We will start with parabolas that are situated with the vertex at the origin and that have a vertical or horizontal axis of symmetry.

GEOMETRIC DEFINITION OF A PARABOLA

A **parabola** is the set of all points in the plane that are equidistant from a fixed point F (called the **focus**) and a fixed line l (called the **directrix**).





PARABOLA WITH VERTICAL AXIS

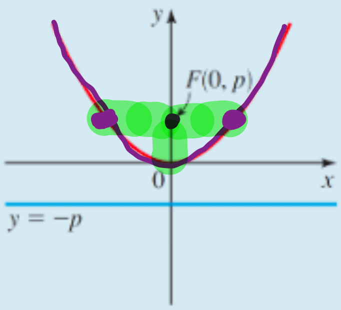
The graph of the equation

$$\boxed{x^2 = 4py}$$

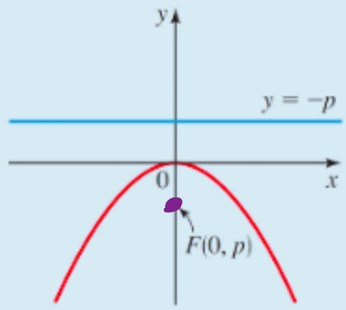
is a parabola with the following properties.

VERTEX	$V(0, 0)$
FOCUS	$F(0, p)$
DIRECTRIX	$y = -p$

The parabola opens upward if $p > 0$ or downward if $p < 0$.



$x^2 = 4py$ with $p > 0$



$x^2 = 4py$ with $p < 0$

PARABOLA WITH HORIZONTAL AXIS

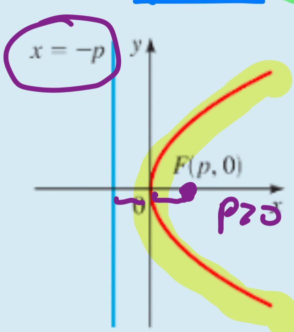
The graph of the equation

$$y^2 = 4px$$

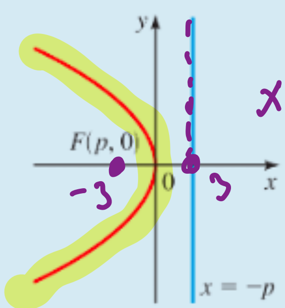
is a parabola with the following properties.

VERTEX	$V(0, 0)$
FOCUS	$F(p, 0)$
DIRECTRIX	$x = -p$

The parabola opens to the right if $p > 0$ or to the left if $p < 0$.



$y^2 = 4px$ with $p > 0$



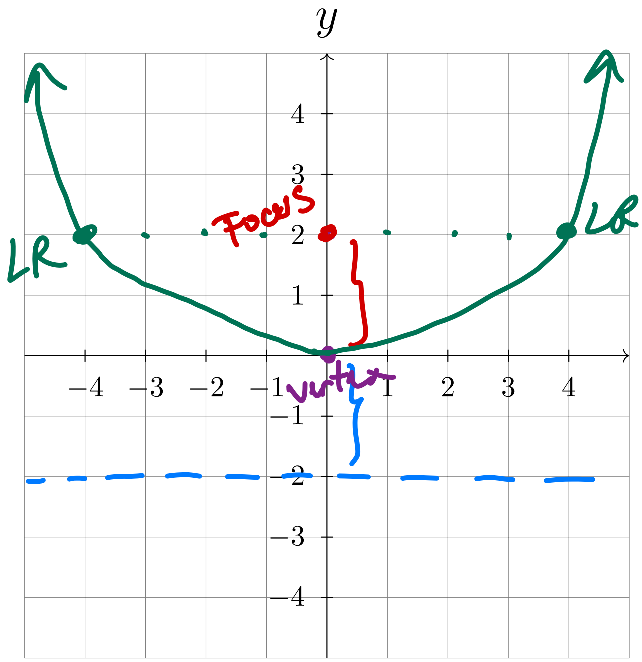
$x = -(-3) = 3$

$y^2 = 4px$ with $p < 0$

Lactus rectum endpoints: are the points on the parabola that $2|p|$ units from the focus.

Note: The parabola always opens towards the focus and away from the directrix!!!

Example 1: Find an equation for the parabola with vertex $V(0,0)$ and focus $F(0,2)$, and sketch its graph.



equation:

$x^2 = 4py$

$p = 2 > 0$ opens up

Loctus rectum:

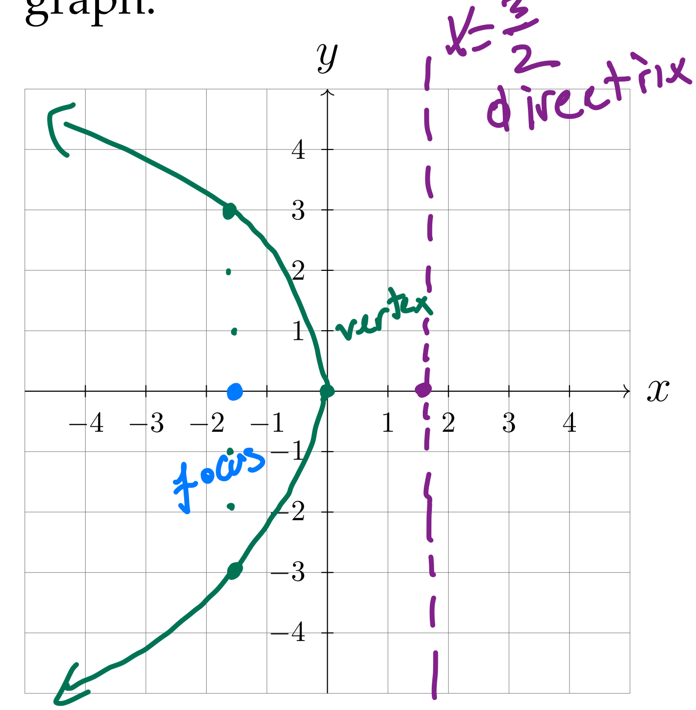
$2|p| = 2(2) = 4$ units from the focus.

directrix $y = -2$

Ans: $x^2 = 4 \cdot 2y \Rightarrow x^2 = 8y$

$y = \frac{1}{8}x^2$

Example 2: A parabola has the equation $6x + y^2 = 0$.
Find the focus, focal diameter and directrix of the parabola and draw the graph.



$6x + y^2 = 0$

$y^2 = -6x$

$y^2 = 4px$

Solve: $4p = -6$

$p = -\frac{6}{4} \Rightarrow p = -\frac{3}{2}$

parabola opens to the left.

need LR $2|p| = 2\left|-\frac{3}{2}\right| =$

$= \cancel{2} \cdot \frac{3}{\cancel{2}} = 3$

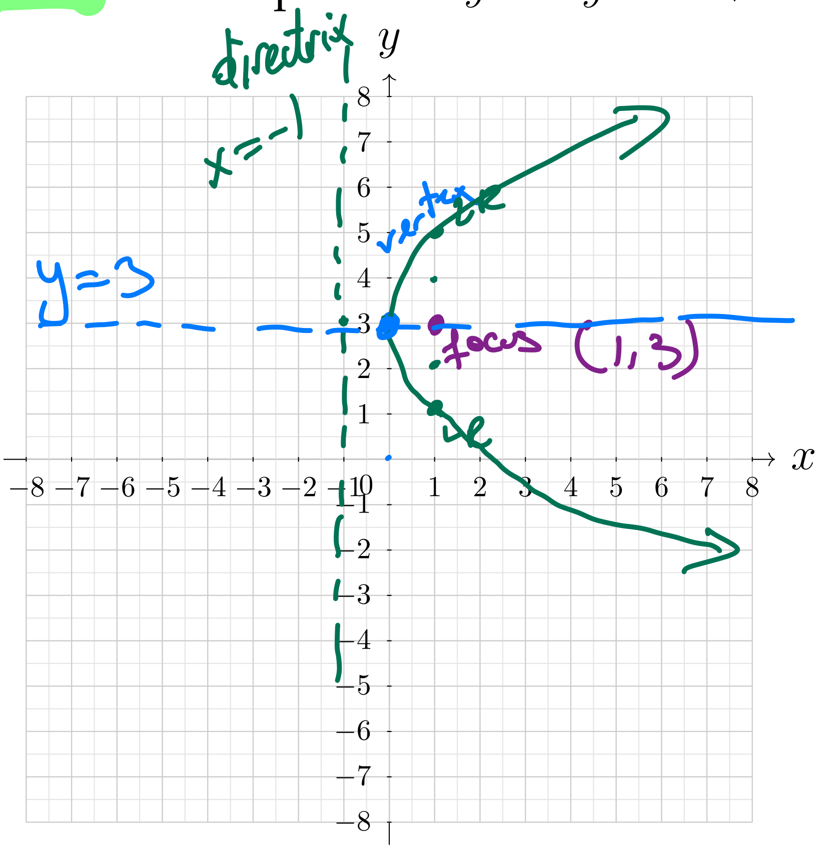
focal diameter : $4|p| = 4\left(\frac{3}{2}\right) = 6$

The following summarizes the equation and features of hyperbolas centered at (h, k).

$x^2 = 4py \xrightarrow{(h,k)} (x-h)^2 = 4p(y-k)$

$y^2 = 4px \longrightarrow (y-k)^2 = 4p(x-h)$

Example 3: Identify the vertex, focus, directrix, axis of symmetry ($y=3$), lactus rectum of the parabola $y^2 - 6y - 4x + 9 = 0$. Draw the parabola.



$y^2 - 6y - 4x + 9 = 0$

$y^2 - 6y = 4x - 9$

$y^2 - 6y + 9 = 4x - 9 + 9$

$(y-3)^2 = 4(x-0)$

$(h,k) = (0,3)$ vertex

$p = 1 > 0$ opens to the right.

LR : $2|p| = 2|1| = 2$