Mth 146: Day 19
11.10 Taylor and Maclaurin Series

1 Part 1: New Concepts & Worked Examples

Which Functions Have Power Series?

In *Representations of Functions as Power Series* we could only find power series for functions that come from the _____. Today we answer two bigger questions:

- **Which functions have power series representations?**
- **How can we find such representations?**

We will see that some of the most important functions in calculus, such as e^x and $\sin x$, can be written as power series.

Suppose f can be represented by a power series

$$\textbf{(1)} \qquad f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \quad |x-a| < R$$

What must the coefficients c_n be? Differentiate term by term and plug in $x = a$ each time:

$$f(x) = c_0 + c_1(x-a) + \cdots \quad \Rightarrow \quad f(a) = _____$$

$$f'(x) = c_1 + 2c_2(x-a) + \cdots \quad \Rightarrow \quad f'(a) = _____$$

$$f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + \cdots \quad \Rightarrow \quad f''(a) = _____$$

$$f'''(x) = 2 \cdot 3c_3 + \cdots \quad \Rightarrow \quad f'''(a) = _____$$

The pattern: $f^{(n)}(a) =$ _____. Solving for c_n gives the formula in the box below. (We use the conventions $0! = 1$ and $f^{(0)} = f$.)

Theorem. If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = _____$$

So *if* f has a power series expansion at a, it must be of the form

Taylor series of f at a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= _____$$

This is called the **Taylor series of the function f at a** (or *about* a, or *centered at* a). The special case $a =$ _____ comes up so often that it gets its own name:

Maclaurin series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \underline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

Notes:

- Finding the Taylor series of f **does not guarantee** that the series _____. There are functions that are not equal to the sum of their Taylor series.
- A power series representation at a is _____: **no matter how we find it, it is the Taylor series.** So every series we found in *Representations of Functions as Power Series* is a Taylor series!

Example 1. We know that $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ for $|x| < 1$. Confirm that this is the Maclaurin series of $f(x) = \dfrac{1}{1-x}$ by computing $c_n = \dfrac{f^{(n)}(0)}{n!}$.

Example 2. For the function $f(x) = e^x$, find the Maclaurin series and its radius of convergence.

When Is a Function Equal to Its Taylor Series?

The partial sums of a Taylor series are polynomials:

The n**th-degree Taylor polynomial of** f **at** a is

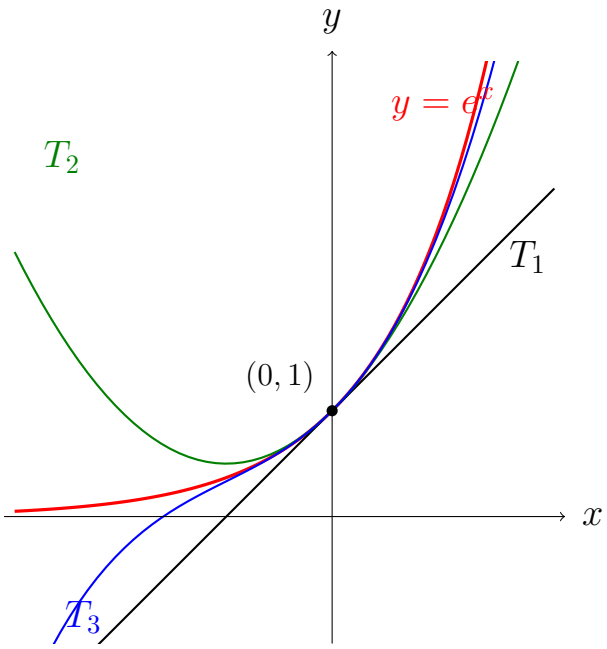
$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$= _____$$

For $f(x) = e^x$ at $a = 0$, Example 2 gives

$$T_1(x) = _____ \qquad T_2(x) = _____$$

$$T_3(x) = _____$$



$y = e^x$ (red) with its Taylor polynomials T_1, T_2, T_3 at $a = 0$

As n increases, $T_n(x)$ appears to approach e^x. In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n\to\infty} _____$$

The difference between f and its Taylor polynomial is called the **remainder**:

$$R_n(x) = _____ \qquad \text{so that} \qquad f(x) = T_n(x) + R_n(x).$$

Theorem. If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a, and if

$$\lim_{n\to\infty} R_n(x) = ____ \qquad \text{for } |x - a| < R,$$

then f is equal to the sum of its Taylor series on the interval $|x - a| < R$.

To show $R_n(x) \to 0$ we usually use:

Taylor's Inequality. If $\left|f^{(n+1)}(x)\right| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies

$$|R_n(x)| \leq _____ \qquad \text{for } |x-a| \leq d$$

A useful fact when applying these theorems:

$$\lim_{n\to\infty} \frac{x^n}{n!} = _____ \qquad \text{for every real number } x$$

This is true because the series $\sum \dfrac{x^n}{n!}$ converges for all x (Example 2), so its nth term must go to 0.

Example 3. Show that e^x is equal to the sum of its Maclaurin series.

$$e^x = _____ \qquad \text{for all } x$$

In particular, putting $x = 1$ gives **the number e as an infinite series**:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = _____$$

Example 4. Find the Taylor series for $f(x) = e^x$ at $a = 2$.

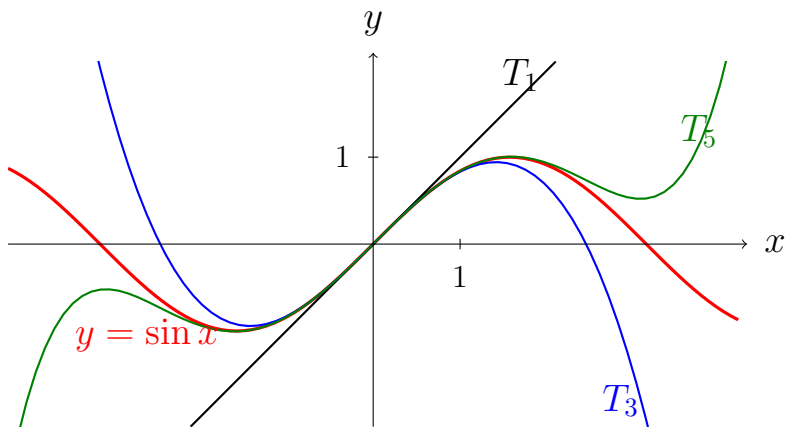
We now have **two** power series for e^x. The Maclaurin series is better for x near _____, and the Taylor series at $a = 2$ is better for x near _____.

Example 5. Find the Maclaurin series for $\sin x$ and show that it represents $\sin x$ for all x.

The Maclaurin polynomials of $\sin x$ are

$$T_1(x) = _____ \qquad T_3(x) = _____$$

$$T_5(x) = _____$$



$y = \sin x$ (red) with its Maclaurin polynomials T_1, T_3, T_5

Notice that as n increases, $T_n(x)$ becomes a _____ to $\sin x$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$= \underline{\qquad\qquad\qquad\qquad} \quad \text{for all } x$$

Example 6. Find the Maclaurin series for $\cos x$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$= \underline{\qquad\qquad\qquad\qquad} \quad \text{for all } x$$

Fun fact: The Maclaurin series for e^x, $\sin x$, and $\cos x$ were discovered by Newton. They tell us that we know *everything* about each of these functions if we know all of its derivatives at the single number 0.

Example 7. Represent $f(x) = \sin x$ as the sum of its Taylor series centered at $\dfrac{\pi}{3}$.

Use the Maclaurin series for x near _____ and the Taylor series at $\frac{\pi}{3}$ for x near _____.

Example 8. Find the Maclaurin series for $f(x) = (1+x)^k$, where k is any real number.

This series is called the **binomial series**. Its coefficients are the **binomial coefficients**:

$$\binom{k}{n} = _____$$

The Binomial Series. If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

$$= _____$$

Notes:

- If k is a **positive integer**, the series _____ (the terms are eventually 0), and we get the ordinary Binomial Theorem.

- Whether the series converges at the endpoints ± 1 depends on k: it converges at 1 if $-1 < k \leq 0$ and at both endpoints if $k \geq 0$.

Example 9. For the function $f(x) = \dfrac{1}{\sqrt{4-x}}$, find the Maclaurin series and its radius of convergence.

Table of Important Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = _____ \qquad R = ____$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = _____ \qquad R = ____$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad R = ____$$

$$= _____$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad R = ____$$

$$= _____$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad R = ____$$

$$= _____$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \qquad R = ____$$

$$= _____$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \qquad R = ____$$

$$= _____$$

Because a power series representation is unique, we can build new Taylor series by manipulating the ones in the table instead of computing $f^{(n)}(a)$. Just like in *Representations of Functions as Power Series*, we can:

- replace x by an expression of the form _____,
- multiply or divide the series by such an expression,
- differentiate or integrate _____,
- and add, subtract, multiply, or divide Taylor series.

Example 10. Find the Maclaurin series for (a) $f(x) = x\cos x$ and (b) $f(x) = \ln(1 + 3x^2)$.

Example 11. Find the function represented by the power series $\sum_{n=0}^{\infty}(-1)^n\frac{2^n x^n}{n!}$.

Example 12. Find the sum of the series
$\frac{1}{1\cdot 2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\cdots$.

2 Part 2: Now You Try It

Use the definition of a Taylor series or the table of important Maclaurin series, as indicated. Final answers are given as footnotes at the bottom of the page — only the final answer, not the steps, so use them to check your work rather than replace it.

1. Use the definition to find the Maclaurin series for $f(x) = e^{5x}$. What is the radius of convergence?
   1

2. Use the definition to find the Taylor series for $f(x) = \ln x$ centered at $a = 2$. What is the radius of convergence?
   2

[1] Answer: $\sum_{n=0}^{\infty} \frac{5^n x^n}{n!}$; $R = \infty$.

[2] Answer: $\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\,2^n}(x-2)^n$; $R = 2$.

3. Use the definition to find the Taylor series for $f(x) = \cos x$ centered at $a = \pi$.

[3]

4. Use the binomial series to find the Maclaurin series for $f(x) = \dfrac{1}{(1+x)^3}$. What is the radius of convergence?

[4]

[3] Answer: $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\pi)^{2n}}{(2n)!}$; $R = \infty$.

[4] Answer: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)}{2} x^n = 1 - 3x + 6x^2 - 10x^3 + \cdots$; $R = 1$.

3 Part 3: Reflection

For each item below, rate your comfort level and jot a quick note: What clicked? What's still fuzzy? What would help?

1. **Use the formula $c_n = \dfrac{f^{(n)}(a)}{n!}$ to find a Taylor or Maclaurin series.** 1 2 3 4 5

2. **Write the nth-degree Taylor polynomial $T_n(x)$ and explain how it approximates f.** 1 2 3 4 5

3. **Recall the important Maclaurin series (e^x, $\sin x$, $\cos x$, $\tan^{-1} x$, $\ln(1+x)$, $(1+x)^k$) and their radii of convergence.** 1 2 3 4 5

4. **Build new series from the table by substituting, multiplying, differentiating, or integrating.** 1 2 3 4 5

5. **Recognize a series as a known function to find its sum.** 1 2 3 4 5